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CORRELATION OF RADIATION FIELD PATTERNS WITH CIRCULARY SYMMETRIC APERTURE DISTRIBUTION.

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CORRELATION OF RADIATION FIELD PATTERNS WITH CIRCULARLY SYMMETRIC APERTURE DISTRIBUTIONS

By

Dave Johnston, Jr. [1915-]

An essay submitted to the Advisory Board of the School of Engineering of The Johns Hopkins University in conformity with the requirement for the degree of Master of Engineering.

Baltimore

1949

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Acknowledgment

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Dedication

This investigation and analysis is dedicated to my wife for without her encouragement this work would never have been completed.

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Chapter I

Introduction

1.1 The problem of diffraction patterns or, as they are often called, radiation patterns, has long enjoyed the attention of both experimental and theoretical workers. This has been particularly true in recent years and considerable effort has been expended to design and develop antenna feeds and reflectors or focusing elements for microwave transmission which would produce distant fields of certain prescribed characteristics. Most of the work has been experimental for, while it is possible to determine accurately a distant field pattern from a known flux distribution over an aperture (1) (2)**, the converse has not been true.

1.2 The problem of relating analytically a known distant field pattern to its source distribution over an aperture has received the attention of only a few investigators. While their work has thrown considerable light on the nature of the problem, their results have not been generally applicable. R. C. Spencer (3) has investigated the Fourier Transform method in considerable detail. Solutions may be obtained by this method in certain cases but one must proceed with caution in extending the limits of integration to infinity. Another approach to the problem has been made by P. M. Woodward and J. D. Lawson (4) in a study of the two dimensional problem. In this study the limits of the aperture are extended to infinity in only one dimension.

1.3 In this paper an analytical method for determining the amplitude distribution of electric and magnetic field vectors over a finite aperture

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**Numbers in parenthesis refer to references listed in Bibliography.

is developed. The method is based on direct integration of the Maxwell field equations and the fact that uniform phase amplitude distributions of plane polarized electric field vectors and their corresponding aperture distributions may be added scalarly at each point in space and over the aperture. It is rigorous and requires no assumptions regarding aperture limits. The only restriction placed on the method from a practical point of view is that the postulated distant field pattern for which the source distribution is sought, must be of such a nature that it may be set up by circularly symmetric, plane polarized waves over the illuminated aperture. A relatively simple means of checking a postulated pattern to determine if it meets this requirement is also developed.

1.4 The integration of Maxwell's field equations has been developed by Stratton in collaboration with Dr. L. J. Chu from a method proposed by Kottler (5). The first part of this paper, Chapter II, is devoted to a summary of Stratton's vector solution of these equations leading up to and including the solution for diffraction or radiation from a surface with discontinuous illumination, such as a finite aperture.

1.5 The analysis is further divided into four parts, presented in Chapters III through VI. Chapter III presents an investigation into the use of Stratton's equations for determining a distant field pattern from a known aperture distribution. The solution for the case of constant amplitude distribution over the aperture is exact and contains the small, generally neglected, component of energy flow normal to the direction of propagation. The results of this analysis have been used throughout the remainder of the paper to provide the general characteristics required of the distant field. In particular, the $\underline{G}$ function is contained in all assumed distant electric field vector patterns and provides the necessary

vector direction to the $\underline{E}$ component of the field in addition to the time & range factor: $\frac{e^{+i(\omega t - kR)}}{R}$

1.6 Chapter IV presents the solution for $I(Z^2)$ encountered in the integral equation** which always arises from circularly symmetric, plane polarized aperture distributions. In theory, the solution is limited to a small class of $I(Z^2)$ functions (aperture distributions) but in practice the restrictions are of little significance.

1.7 Chapter V presents a development similar to the method given in Chapter IV. This development, however, throws considerable light on the types of field patterns which are theoretically possible. It also provides a means of solving for the aperture distribution, $I(Z^2)$, but in general it is not as neat a method as that developed in Chapter IV. It is slightly more general and could be of use in some cases where the method presented in Chapter IV fails.

1.8 Chapter VI presents a very special type solution for the aperture distribution where $I(Z)$ is a function of $\hat{\rho}$ alone and is independent of "a". It is simple and direct but very limited in use.

1.9 Conclusions are contained in Chapter VII.

1.10 A list of symbols used with definitions and a table of vector identities are contained in Appendices I and II. Curves of required aperture distributions to give certain distant space energy patterns are contained in Appendix III. Appendix IV contains a short discussion on one special property of determinants.

**

$$\underline{E}(x) = \underline{G}\, a^2 \int_0^1 J_0(xZ)\, Z\, I(Z^2)\, dZ$$

Chapter II

Direct Integration of Maxwell's Field Equations

2.1 The solution to Maxwell's equations given here is the solution developed by J. A. Stratton (1) and is contained in detail in his Electromagnetic Theory, sections 8.14 and 8.15.** The following development has been modified slightly to better serve the purpose of this paper.

2.2 We shall postulate that at every point in space the electric and magnetic field vectors are subject to Maxwell's field equations. Further, let us assume that the field equations contain the time only as a factor $e^{+i\omega t}$ and write the field equations in the form:

$$\underline{\nabla}\times\underline{E} - i\omega\mu\underline{H} = -\underline{J}^* \tag{2.01}$$

$$\underline{\nabla}\times\underline{H} + i\omega\varepsilon\underline{E} = \underline{J} \tag{2.02}$$

$$\underline{\nabla}\cdot\underline{H} = \frac{1}{\mu}\rho^* \qquad (2.03)\ (6)$$

$$\underline{\nabla}\cdot\underline{E} = \frac{1}{\varepsilon}\rho \tag{2.04}$$

where $\underline{J}^*$ and ρ^* are fictitious densities of "magnetic current" and "magnetic charge" which to the best of our knowledge have no physical existence. Both the real and fictitious currents and charges are related by the equations of continuity

$$\underline{\nabla}\cdot\underline{J} - i\omega\rho = 0 \tag{2.05}$$

$$\underline{\nabla}\cdot\underline{J}^* - i\omega\rho^* = 0 \tag{2.06}$$

** Stratton uses the expression e^{+ikR} for a positive traveling wave. This is a matter of controversy and the writer, in conformity with engineering practices, prefers to use e^{-ikR}.

which may be readily shown to be satisfied by the field equations by taking the divergence of (2.01) and (2.02).

2.3 The vectors $\underline{E}$ and $\underline{H}$ satisfy**

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) - k^2 \underline{E} = i\omega\mu \underline{J} - \underline{\nabla} \times J^* \tag{2.07}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) - k^2 \underline{H} = i\omega\varepsilon \underline{J}^* + \underline{\nabla} \times \underline{J} \tag{2.08}$$

where $k^2 = \omega^2 \varepsilon\mu$ since

$$\underline{\nabla} \times \underline{E} - i\omega\mu \underline{H} = -\underline{J}^* \tag{2.01}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) - i\omega\mu\, \underline{\nabla} \times \underline{H} = -\underline{\nabla} \times \underline{J}^*$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) - i\omega\mu (\underline{J} - i\omega\varepsilon \underline{E}) = -\underline{\nabla} \times \underline{J}^*$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) - \omega^2 \varepsilon\mu \underline{E} = i\omega\mu \underline{J} - \underline{\nabla} \times \underline{J}^* \tag{2.09}$$

and

$$\underline{\nabla} \times \underline{H} + i\omega\varepsilon \underline{E} = \underline{J} \tag{2.02}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) + i\omega\varepsilon\, \underline{\nabla} \times \underline{E} = \underline{\nabla} \times \underline{J}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) + i\omega\varepsilon (-\underline{J}^* + i\omega\mu \underline{H}) = \underline{\nabla} \times \underline{J}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{H}) - \omega^2 \varepsilon\mu \underline{H} = i\omega\varepsilon \underline{J}^* + \underline{\nabla} \times \underline{J} \tag{2.10}$$

2.4 A direct proof of the desired result can be obtained by applying the vector analogue of Green's Theorem to the field equations. Let V be a closed region of space bounded by a regular surface S, and let $\underline{P}$ and $\underline{Q}$ be two vector functions of position which together with their first and second derivatives are continuous throughout V and on the surface S, then, applying the divergence theorem to the vector $\underline{P} \times (\underline{\nabla} \times \underline{Q})$

$$\int_V \underline{\nabla} \cdot \left[\underline{P} \times (\underline{\nabla} \times \underline{Q}) \right] dv = \int_S \left[\underline{P} \times (\underline{\nabla} \times \underline{Q}) \right] \cdot \underline{n}\, da \tag{2.11}$$

[**] A table of vector identities is contained in Appendix II.

The integrand of the volume integral may be expanded to

$$\int_V \left[\underline{\nabla}\times\underline{P}\cdot\underline{\nabla}\times\underline{Q}-\underline{P}\cdot\underline{\nabla}\times(\underline{\nabla}\times\underline{Q})\right] dv=\int_S \Big[\underline{P}\times(\underline{\nabla}\times\underline{Q})\cdot\underline{n}\ da \qquad (2.12)$$

By simply interchanging $\underline{P}$ and $\underline{Q}$

$$\int_V \left[\underline{\nabla}\times\underline{Q}\cdot\underline{\nabla}\times\underline{P}-\underline{Q}\cdot\underline{\nabla}\times(\underline{\nabla}\times\underline{P})\right] dv=\int_S \Big[\underline{Q}\times(\underline{\nabla}\times\underline{P})\cdot\underline{n}\ da \qquad (2.13)$$

and subtracting (2.13) from (2.14) we have:

$$\int_V \left[\underline{Q}\cdot\underline{\nabla}\times(\underline{\nabla}\times\underline{P})-\underline{P}\cdot\underline{\nabla}\times(\underline{\nabla}\times\underline{Q})\right] dv$$

$$=\int_S \left[\underline{P}\times(\underline{\nabla}\times\underline{Q})-\underline{Q}\times(\underline{\nabla}\times\underline{P})\right]\cdot\underline{n}\ da \qquad (2.14)$$

2.5 In equation (2.14)

let $\underline{P}\equiv\underline{E}$

$\underline{Q}\equiv\phi\ \underline{a}$ where $\underline{a}$ is a unit vector in an arbitrary direction and

$$\phi=\frac{e^{-ikr}}{r}\quad ;\quad \underline{\nabla\phi}=-\phi\left(ik+\frac{1}{r}\right)\underline{r_0} \qquad (2.15)$$

$$\underline{\nabla}\cdot\underline{\nabla\phi}=-k^2\phi \qquad (2.16)$$

We have the following identities:

(1) $\underline{\nabla}\times\underline{Q}=\underline{\nabla}\times\phi\,\underline{a}=\underline{\nabla\phi}\times\underline{a}=\phi\,\underline{\nabla}\times\underline{a}$

$$\therefore\ \underline{\nabla}\times\underline{Q}=\underline{\nabla\phi}\times\underline{a}\qquad \text{since}\quad \underline{\nabla}\times\underline{a}=0 \qquad (2.17)$$

(2) $\underline{\nabla}\times\underline{\nabla}\times\underline{Q}=\underline{\nabla\phi}(\underline{\nabla}\cdot\underline{a})-\underline{a}(\underline{\nabla}\cdot\underline{\nabla\phi})+(\underline{a}\cdot\underline{\nabla})\underline{\nabla\phi}-(\underline{\nabla\phi}\cdot\underline{\nabla})\,\underline{a}$

$=\underline{a}\,k^2\phi+(\underline{a}\cdot\underline{\nabla})\,\underline{\nabla\phi}$

$\underline{\nabla}\cdot\underline{a}=0$ and $(\underline{\nabla\phi}\cdot\underline{\nabla})\,\underline{a}=0$ since $\underline{a}$ is a constant

$(\underline{a}\cdot\underline{\nabla})\,\underline{\nabla\phi}=\underline{\nabla}(\underline{a}\cdot\underline{\nabla\phi})-(\underline{\nabla\phi}\cdot\underline{\nabla})\underline{a}+\underline{a}\times(\underline{\nabla}\times\underline{\nabla\phi})$

$+\ \underline{\nabla\phi}\times(\underline{\nabla}\times\underline{a})=\underline{\nabla}(\underline{a}\cdot\underline{\nabla\phi})$ since

$$(\nabla\phi\cdot\underline{\nabla})\underline{a}=0\ , \quad \underline{\nabla}\times\nabla\underline{\phi}=0 , \quad \underline{\nabla}\times\underline{a}=0$$

$$\therefore\ \underline{\nabla}\times(\underline{\nabla}\times\underline{Q})=\underline{a}k^2\phi+\underline{\nabla}(\underline{a}\cdot\underline{\nabla}\underline{\phi}) \qquad (2.18)$$

$$(3) \quad (\underline{\nabla}\times\underline{P}) = (\underline{\nabla}\times\underline{E}) \qquad (2.19)$$

$$(4) \quad \underline{\nabla}\times(\underline{\nabla}\times\underline{P})=\underline{\nabla}\times(\underline{\nabla}\times\underline{E})=k^2\underline{E}+i\omega\mu\underline{J}-\underline{\nabla}\times\underline{J}^* \qquad (2.20)$$

Substituting equations (2.17), (2.18), (2.19) and (2.20) in equation (2.14) we have:

$$\int_V\left\{\phi\underline{a}\cdot\left[k^2\underline{E}+i\omega\mu\underline{J}-\underline{\nabla}\times\underline{J}^*\right]-\underline{E}\cdot\left[\underline{a}k^2\phi+\underline{\nabla}(\underline{a}\cdot\underline{\nabla}\underline{\phi})\right]\right\}dv$$
$$=\int_S\left[\underline{E}\times(\underline{\nabla}\underline{\phi}\times\underline{a})-\phi\,\underline{a}\times(\underline{\nabla}\times\underline{E})\right]\cdot\underline{n}\,da \qquad (2.21)$$

The integral over the volume may be written

$$\underline{a}\cdot\int_V(i\omega\mu\underline{J}\phi-\underline{\nabla}\times\underline{J}^*\phi)\,dv-\int_V\underline{E}\cdot\underline{\nabla}(\underline{a}\cdot\underline{\nabla}\underline{\phi})\,dv$$
$$=a\cdot\int_V(i\omega\mu\underline{J}\phi-\underline{\nabla}\times\underline{J}^*\phi)\,dv+\int_V(\underline{a}\cdot\underline{\nabla}\underline{\phi})(\underline{\nabla}\cdot\underline{E})\,dv$$
$$-\int_V\underline{\nabla}\cdot\left[(\underline{a}\cdot\underline{\nabla}\underline{\phi})\underline{E}\right]dv=\underline{a}\cdot\int_V\Big[i\omega\mu\underline{J}\phi-\underline{\nabla}\times\underline{J}^*\phi$$
$$+\underline{\nabla}\underline{\phi}(\underline{\nabla}\cdot\underline{E})\Big]\,dv-\underline{a}\cdot\int_S(\underline{E}\cdot\underline{n})\,\underline{\nabla}\underline{\phi}\,da \qquad (2.22)$$

The integral over the surface in equation (2.20) may be written

$$\int_S\Big\{\left[(\underline{a}\cdot\underline{E})(\underline{\nabla}\phi\cdot\underline{n})-(\underline{E}\cdot\underline{\nabla}\underline{\phi})(\underline{a}\cdot\underline{n})\right]-\phi\Big[(\underline{a}\cdot\underline{E})(\underline{\nabla}\cdot\underline{n})$$
$$+(\underline{a}\cdot\underline{\nabla})(\underline{E}\cdot\underline{n})\Big]\Big\}\,da$$
$$=\underline{a}\cdot\int_S\left\{(\underline{n}\cdot\underline{\nabla}\underline{\phi})\underline{E}-(\underline{E}\cdot\underline{\nabla}\underline{\phi})\underline{n}-\phi\left[(\underline{n}\cdot\underline{\nabla})\underline{E}-(\underline{E}\cdot\underline{n})\,\underline{\nabla}\right]\right\}da$$
$$=\underline{a}\cdot\int_S\left\{\underline{\nabla}\underline{\phi}\times(\underline{E}\times\underline{n})+\phi\left[\underline{n}\times(\underline{\nabla}\times\underline{E})\right]\right\}da$$
$$=\underline{a}\cdot\int_S(\underline{n}\times\underline{E})\times\underline{\nabla}\underline{\phi}+\phi\left[\underline{n}\times(i\omega\mu\underline{H}-J^*)\right]\Big\}\,da \qquad (2.23)$$

Combining equations (2.22) and (2.23) and since $\underline{a}$ is arbitrary

$$\int_V \left(i\omega\mu \underline{J}\phi - \underline{\nabla}\times\underline{J}^*\phi + \underline{\nabla}\phi\frac{1}{\varepsilon}\rho\right) dv = \int_S \Big[i\omega\mu(\underline{n}\times\underline{H})\phi + (\underline{n}\times\underline{E})\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E})\,\underline{\nabla\phi} - \underline{n}\times\underline{J}^*\Big]\, da \qquad (2.24)$$

The identity

$$\int_V \underline{\nabla}\times\phi\,\underline{J}^* dv = \int_S \underline{n}\times\underline{J}^*\phi\, da$$

$$\int_V \Big[(\underline{\nabla}\times\underline{J}^*)\phi - \underline{J}^*\times\underline{\nabla\phi}\Big] dv = \int_S \underline{n}\times\underline{J}^*\phi\, da \quad \text{or}$$

$$\int_V \underline{\nabla}\times\underline{J}^*\phi\, dv = \int_S \underline{n}\times\underline{J}^*\phi\, da + \int_V \underline{J}^*\times\underline{\nabla\phi}\, dv$$

reduces equation (2.24) to

$$\int_V \left(i\omega\mu\underline{J}\phi - \underline{J}^*\times\underline{\nabla\phi} + \frac{1}{\varepsilon}\rho\,\underline{\nabla\phi}\right) dv = \int_S \Big[i\omega\mu(\underline{n}\times\underline{H})\phi + (\underline{n}\times\underline{E})\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E})\,\underline{\nabla\phi}\Big]\, da \qquad (2.25)$$

2.6 The validity of this relation has been established for regions within which both $\underline{P}\equiv\underline{E}$ and $\underline{Q}\equiv\phi\,\underline{a}$ are continuous and possess continuous first and second derivatives.. $\underline{Q}$ however, has a singularity at $r = 0$ and consequently this point must be excluded. Let x', y', z' be the coordinate of an interior point and let a sphere of radius r_1 be circumscribed about the point x', y', z' its normal, $\underline{n}$, directed out of V and consequently radially toward the center. The area of the sphere vanishes with the radius as $4\pi r_1^2$ and since $\underline{\nabla\phi} = -\phi\left(ik + \frac{1}{r}\right)\underline{r_0} = \phi\left(ik + \frac{1}{r}\right)\underline{n}$ and $(\underline{n}\times\underline{E})\times\underline{n} + (\underline{n}\cdot\underline{E})\underline{n} = \underline{E}$ the contribution of the spherical surface to the right hand side of equation (2.25) reduces to $4\pi E(x', y', z')$

The value of $\underline{E}$ at any interior point of V is, therefore:

$$\underline{E}(x', y', z') = \frac{1}{4\pi}\int_V \left(i\omega\mu\,\underline{J}\phi - \underline{J}^*\times\underline{\nabla\phi} + \frac{1}{\varepsilon}\rho\,\underline{\nabla\phi}\right) dv - \frac{1}{4\pi}\int_S \Big[i\omega\mu(\underline{n}\times\underline{H})\phi + (\underline{n}\times\underline{E})\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E})\,\underline{\nabla\phi}\Big]\, da \qquad (2.26)$$

An obvious interchange of vectors leads to the corresponding expression for $\underline{H}$.

$$H(x',y',z') = \frac{1}{4\pi}\int_V \left(i\omega\varepsilon\,\underline{J}^*\phi + \underline{J}\times\underline{\nabla\phi} + \frac{1}{\mu}\rho^*\,\underline{\nabla\phi}\right)dv$$

$$+\frac{1}{4\pi}\int_S\left[i\omega\varepsilon(\underline{n}\times\underline{E})\phi - (\underline{n}\times\underline{H})\times\underline{\nabla\phi} - (\underline{n}\cdot\underline{H})\,\underline{\nabla\phi}\right]da \qquad (2.27)$$

2.7 If all currents and charges can be enclosed within a sphere of finite radius, the field is regular at infinity and either side of S may be chosen as its interior.

2.8 Let us suppose now that the charge and current densities are confined to a thin layer at the surface S. As the depth of the layer diminishes, the densities may be increased so that in the limit the volume densities are replaced by surface densities. If the region V contains no charge or current within its interior or on its boundary S, the field at an interior point is

$$\underline{E}(x',y',z') = -\frac{1}{4\pi}\int_S\left[i\omega\mu(\underline{n}\times\underline{H})\phi + (\underline{n}\times\underline{E})\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E})\nabla\phi\right]da$$

and since either side of S may be chosen as its interior

$$\underline{E(x)} = -\frac{1}{4\pi}\int_S\left[i\omega\mu(\underline{n}\times\underline{H})\phi + (\underline{n}\times\underline{E})\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E})\nabla\phi\right]da \qquad (2.28)$$

where $\underline{E(x)}$ is the electric field vector distribution out of the aperture.

2.9 In the case of radiation from an aperture the aperture will be defined as a finite circular area in space of radius "a" over which the source distribution of electromagnetic energy exists. The aperture is shielded in the direction of the negative normal by a perfectly absorbing screen of radius "a" through which electromagnetic energy may not pass. The area over the face of a parabolic reflector closely approximates this definition of an aperture.

2.10 Equation (2.26) holds true only if $\underline{E}$ and $\underline{H}$ are continuous and have continuous first derivatives at all points of S. It cannot, there-

fore, be applied directly to the problem of radiation from an aperture. To obtain the required extension to such cases consider the closed surface S (closed at infinity) to be divided into two zones S_1 and S_2 by a closed contour C, as in Fig. 1.

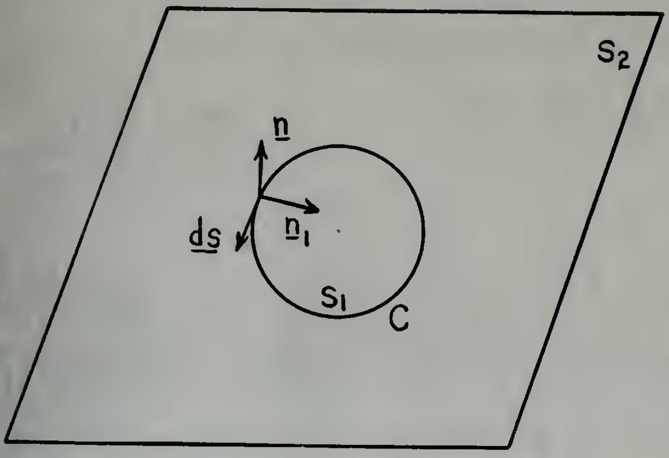

Figure 1

The vectors $\underline{E}$ and $\underline{H}$ and their first derivatives are continuous over S_1 and satisfy the field equations. The same is true for S_2. However, the components of $\underline{E}$ and $\underline{H}$ which are tangential to the surface are subject to a discontinuous change in passing across C. The occurence of such discontinuities can be reconciled with the field equations only by the further assumption of a line distribution of charges or currents about the contour C. This line distribution of sources contributes to the field, and only when it is taken into account do the resultant expressions for $\underline{E}$ and $\underline{H}$ satisfy Maxwell's equations.

2.11 A method of determining a contour distribution consistent with the requirements of the problem was proposed by Kottler (5). A discontinuity in the tangential components of $\underline{E}$ and $\underline{H}$ in passing on the surface from zone S_1 to zone S_2 implies an abrupt change in the surface current density. The termination of a line of current, in turn, can be accounted for according to the equation of continuity by an accumulation of charge on the contour. Let $\underline{ds}$ be an element of length along the contour in the positive direction as determined by the positive normal $\underline{n}$ in Fig. 1. Let $\underline{n}_1$ be a unit vector lying in the surface, normal to both $\underline{n}$ and $\underline{ds}$ and directed into zone 1. Designate the line densities of electric and mag-

netic charge by σ and σ^*. The equations (2.05) and 2.06), when applied to surface currents become

$$\underline{n}_1 \cdot (\underline{K}_1 - \underline{K}_2) = i\omega\sigma ; \qquad \underline{n}_1 \cdot (\underline{K}_1^* - \underline{K}_2^*) = i\omega\sigma^*$$

where $K \equiv$ electric current surface density

$K^* \equiv$ magnetic current surface density

$\underline{K} = -\underline{n} \times \underline{H}$; $\qquad \underline{K}^* = \underline{n} \times \underline{E}$

Hence:

$$i\omega\sigma = \underline{n}_1 \cdot (\underline{n} \times \underline{H}_2 - \underline{n} \times \underline{H}_1) = (\underline{H}_2 - \underline{H}_1) \cdot (\underline{n}_1 \times \underline{n})$$

$$i\omega\sigma^* = \underline{n}_1 \cdot (\underline{n} \times \underline{E}_1 - \underline{n} \times \underline{E}_2) = -(\underline{E}_2 - \underline{E}_1) \cdot (\underline{n}_1 \times \underline{n})$$

$$\underline{n}_1 \times \underline{n} = \underline{ds} \qquad (\text{Figure } 1)$$

2.12 For radiation from an aperture S_2 represents an opaque screen and over it $\underline{E}_2$ and $\underline{H}_2$ are everywhere zero. Therefore, the field at any point on the shadow side is from equation (2.26)

$$\underline{E(x)} = -\frac{1}{i\omega\varepsilon}\frac{1}{4\pi}\oint_c \underline{\nabla\phi}\,(\underline{H}_1 \cdot \underline{ds}) - \frac{1}{4\pi}\int_s \Big[i\omega\mu\,(\underline{n} \times \underline{H}_1)\phi + (\underline{n} \times \underline{E}_1) \times \underline{\nabla\phi} + (\underline{n} \cdot \underline{E}_1)\nabla\phi \Big]\, da \qquad (2.29)$$

and for the magnetic field vector distribution:

$$\underline{H(x)} = \frac{1}{i\omega\mu}\frac{1}{4\pi}\oint_c \underline{\nabla\phi}\,\underline{E}_1 \cdot \underline{ds} + \frac{1}{4\pi}\int_s \Big[i\omega\varepsilon(\underline{n} \times \underline{E}_1)\phi - (\underline{n} \times \underline{H}_1) \times \underline{\nabla\phi} - (\underline{n} \cdot \underline{H})\underline{\nabla\phi} \Big]\, da \qquad (2.30)$$

Equations (2.29) and (2.30) will be used as the initial relations in the analytical developments which follow.

Chapter III

Determination of the Distant Field

3.1 It has been shown by Stratton and Chu (Chapter I of this paper) that the electric field vector resulting from radiation through an aperture may be expressed as

$$4\pi\, \underline{E}(\underline{x}) = -\frac{1}{i\omega\varepsilon}\oint_C \nabla\phi\,(\underline{H}_1\cdot ds) - \int_S \Big[\, i\omega\mu\,(\underline{n}\times\underline{H}_1)\,\phi + (\underline{n}\times\underline{E}_1)\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E}_1)\,\underline{\nabla\phi}\Big]\, da \qquad (3.01)^{**}$$

The subscript 1 in this expression refers to area 1 (area over the aperture). Since $\underline{E}$ and $\underline{H}$ are zero throughout area 2 (area in plane of aperture outside the aperture) $\underline{E}_1$ and $\underline{H}_1$ will simply be written as $\underline{E}$ and $\underline{H}$. In addition, the development will be confined to circularly symmetric, plane polarized $\underline{E}$ and $\underline{H}$ distributions so that $\underline{E} = \underline{I}(\hat{\rho}, a)$; $\underline{H} = \underline{M}(\hat{\rho}, a)$.

3.2 The purpose of solving equation (3.01) is to determine the vector and range characteristics of $\underline{E}(x)$ resulting from $\underline{E}$ and $\underline{H}$ distributions as assumed above.. This end will be met most easily by assuming a constant amplitude distribution over the aperture and this will be done. At present, however, for development purposes let

$$\underline{E} = I(\hat{\rho}, a)\,\underline{i} \quad \text{and}$$

$$\underline{H} = M(\hat{\rho}, a)\,\underline{j}$$

and let the aperture lie in the $\hat{x}$, $\hat{y}$, plane such that $\underline{n} \equiv \underline{k}$, Fig. 2, page .

**The time factor $e^{+i\omega t}$ is understood to be present in this and similar expressions. It will later be included in the $\underline{G}$ and $\underline{F}$ functions. $\underline{E}$ and $\underline{H}$ are in time phase over the aperture.

3.3 In equation (3.01) consider

$$\oint_C \underline{\nabla\phi}\,(\underline{H}\cdot\underline{ds})$$

Note that for $\underline{H} = M(\hat{\rho},a)\,\underline{j}$ and $\underline{ds}$ in the $\hat{x}$, $\hat{y}$, plane $\underline{H} \cdot \underline{ds}$ will cancel in pairs about the contour C. Hence,

$$\oint_C \underline{\nabla\phi}\,(\underline{H}\cdot\underline{ds}) = 0$$ and

$$4\pi\,\underline{E(x)} = -\int_S \left[i\omega\mu(\underline{n}\times\underline{H})\phi + (\underline{n}\times\underline{E})\times\underline{\nabla\phi} + (\underline{n}\cdot\underline{E})\,\underline{\nabla\phi} \right] da \tag{3.02}$$

Performing the indicated vector multiplication:

$$i\omega\mu(\underline{n}\times\underline{H})\phi = i\omega\mu\left[\underline{k}\times M(\hat{\rho},a)\,\underline{j}\right] = -i\omega\mu\,M(\hat{\rho},a)\,\underline{i}$$

and since $H = \frac{E}{\eta}^{(2)}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}, \quad c = \frac{1}{\sqrt{\mu\varepsilon}} = f\lambda, \quad k = \frac{2\pi}{\lambda}$

we may write

$$i\omega\mu(\underline{n}\times\underline{H})\phi = -i\omega\mu\,M(\hat{\rho},a)\,\underline{i} = -\frac{i\omega\mu}{\eta}\,I(\hat{\rho},a)\,\underline{i} \tag{3.03}$$

$$= -\frac{i\omega}{\sqrt{\mu\varepsilon}}\,I(\hat{\rho},a)\,\underline{i} = -ik\,I(\hat{\rho},a)\,\underline{i}$$

and $$(\underline{n}\times\underline{E})\times\underline{\nabla\phi} = \left[\underline{k}\times I(\hat{\rho},a)\,\underline{i}\right]\times\underline{\nabla\phi}$$

$$= I(\hat{\rho},a)\left[\underline{j}\times\underline{\nabla\phi}\right]$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\,\underline{r_0} = -\frac{e^{-ikr}}{r}\left(ik + \frac{1}{r}\right)\underline{r_0} = -ik\phi\left(1 + \frac{1}{ikr}\right)\underline{r_0}$$

$$(\underline{n}\times\underline{E})\times\nabla\underline{\phi} = -ik\phi\, I(\hat{\rho}, a)\left[1 + \frac{1}{ikr}\right]\left[\underline{j}\times\underline{r_0}\right]$$

$$\underline{r_0} = \sin\theta\cos\varphi\,\underline{i} + \sin\theta\sin\varphi\,\underline{j} + \cos\theta\,\underline{k}$$

$\underline{j}\times\underline{r_0} = \cos\theta\,\underline{i} - \sin\theta\cos\varphi\,\underline{k}$ and

$$(\underline{n}\times\underline{E})\times\underline{\nabla\phi} = -ik\phi\, I(\hat{\rho}, a)\left(1 + \frac{1}{ikr}\right)\left(\cos\theta\,\underline{i} - \sin\theta\cos\varphi\,\underline{k}\right) \tag{3.04}$$

$$(\underline{n}\cdot\underline{E})\,\underline{\nabla\phi} = I(\hat{\rho}, a)\left[\underline{k}\cdot\underline{i}\right]\,\underline{\nabla\phi} = 0 \tag{3.05}$$

Substituting equations (3.03), (3.04) and (3.05) in equation (3.02)

we have:

$$4\pi\,\underline{E}(x) = ik\int_S \phi\, I(\hat{\rho}, a)\left[\underline{i} + \left(1 + \frac{1}{ikr}\right)\left(\cos\theta\,\underline{i} - \sin\theta\cos\varphi\,\underline{k}\right)\right] da \tag{3.06}$$

For the distant field, $r >>> 1$ and $\left(1 + \frac{1}{ikr}\right) \longrightarrow 1$

hence
$$4\pi\,\underline{E(x)} = ik\left[(1+\cos\theta)\,\underline{i} - \sin\theta\cos\varphi\,\underline{k}\right]\int_S \phi\, I(\hat{\rho}, a)\, da \tag{3.07}$$

$$= \frac{2\pi i}{\lambda}\left[(1+\cos\theta)\,\underline{i} - \sin\theta\cos\varphi\,\underline{k}\right]\int_S \frac{e^{-ikr}}{r}\, I(\hat{\rho}, a)\, da \tag{3.08}$$

3.4 From the geometric relations derived in Fig. 2

$$da = \hat{\rho}\, d\hat{\rho}\, d\hat{\varphi}$$

$r = R - \hat{\rho}\sin\theta\cos(\hat{\varphi} - \varphi)$. In this expression $R >>> \hat{\rho}\sin\theta\cos(\hat{\varphi} - \varphi)$

and we may substitute r = R in the denominator of equation (3.08).

This gives $\underline{E}(x) = \frac{2\pi i}{4\pi\lambda}\left\{\left[1+\cos\right]\underline{i} - \sin\theta\,\cos\varphi\,\underline{k}\right\}\left\{\right.$

$$\left.\frac{e^{-ikR}}{R}\int_S I(\hat{\rho},a)\,e^{ik\hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)}\right\}\hat{\rho}\,d\hat{\rho}\,d\hat{\varphi}$$

$$= \frac{\underline{G}}{2\pi}\int_{\hat{\rho}=0}^{a}\int_{\hat{\varphi}=0}^{2\pi} I(\hat{\rho},a)\,e^{ik\hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)}\,\hat{\rho}\,d\hat{\rho}\,d\hat{\varphi} \qquad (3.09)$$

where $\underline{G} = i\pi\frac{e^{+i(\omega t-kR)}}{\lambda R}\left[(1+\cos\theta)\underline{i} - \sin\theta\cos\varphi\,\underline{k}\right]$ (3.10)

3.5 Now, since φ is independent of $\hat{\varphi}$ we may replace $d\hat{\varphi}$ by $d(\hat{\varphi}-\varphi)$ in equation (3.09). Also let $k\hat{\rho}\sin\theta = z$

Equation (3.09) becomes

$$\underline{E}(x) = \frac{\underline{G}}{2\pi}\int_{\hat{\rho}=0}^{a} I(\hat{\rho},a)\,\hat{\rho}\,d\hat{\rho}\int_0^{2\pi} e^{iz\cos(\hat{\varphi}-\varphi)}\,d(\hat{\varphi}-\varphi)$$

From Jahnke and Emde (7), page 149

$$J_n(z) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{iz\cos\varphi}\,e^{in\varphi}\,d\varphi \quad \text{OR}$$

$$2\pi J_0(z) = \int_0^{2\pi} e^{iz\cos\varphi}\,d\varphi$$

$$\therefore\ \underline{E}(x) = \frac{\underline{G}}{2\pi}\int_{\hat{\rho}=0}^{a} I(\hat{\rho},a)\,\hat{\rho}\,2\pi J_0(k\hat{\rho}\sin\theta)\,d\hat{\rho}$$

$$= \underline{G}\int_0^{a} I(\hat{\rho},a)\,\hat{\rho}\,J_0(k\hat{\rho}\sin\theta)\,d\hat{\rho} \qquad (3.11)$$

3.6 To determine the general form of $\underline{E}(x)$ let $I(\hat{\rho},a) = \bar{I}$

i.e. constant amplitude distribution over the aperture then

$$\underline{E}(x) = \underline{G}\,\bar{I}\int_0^{a}\hat{\rho}\,J_0(k\hat{\rho}\sin\theta)\,d\hat{\rho}$$

$$= \underline{G}\,\bar{I}\int_0^{a}\frac{k\hat{\rho}\sin\theta}{k^2\sin^2\theta}\,J_0(k\hat{\rho}\sin\theta)\,d(k\hat{\rho}\sin\theta)$$

$$\underline{E}(x) = \underline{G}\,\overline{I}\,\frac{k\hat{\rho}\sin\theta\, J_1(k\hat{\rho}\sin\theta)}{(k\sin\theta)^2}\Bigg|_{\hat{\rho}=0}^{\hat{\rho}=a} \qquad (7)$$

$$= \underline{G}\,\overline{I}\,a\,\frac{J_1(ka\sin\theta)}{k\sin\theta} = \underline{G}\,\overline{I}\,a^2\,\frac{J_1(ka\sin\theta)}{ka\sin\theta}$$

let $x = ka\sin\theta$ and

$$\underline{E}(x) = \underline{G}\,\overline{I}\,a^2\,\frac{J_1(x)}{x} \tag{3.12}$$

3.7 Similarly, using equation (2.30) we may solve for the magnetic field vector distribution in distant space.

From Stratton $4\pi\underline{H}(x) = \frac{1}{i\omega\mu}\oint_c \underline{\nabla\phi}\,\underline{E}\cdot\underline{ds} + \int_s \Big[i\omega\varepsilon(\underline{n}\times\underline{E})\phi -$

$$(\underline{n}\times\underline{H})\times\underline{\nabla\phi} - (\underline{n}\cdot\underline{H})\underline{\nabla\phi}\Big]\,da \tag{3.13}$$

Again, since $\underline{E} = I(\hat{\rho},a)\underline{i}$ and $\underline{ds}$ lies in the x, y, plane

$\oint_c \underline{\nabla\phi}\,\underline{E}\cdot\underline{ds} \equiv 0$ and

$$4\pi\underline{H}(x) = \int_s \Big[i\omega\varepsilon(\underline{n}\times\underline{E})\phi - (\underline{n}\times\underline{H})\times\underline{\nabla\phi} - (\underline{n}\cdot\underline{H})\underline{\nabla\phi}\Big]\,da \tag{3.14}$$

$$i\omega\varepsilon(\underline{n}\times\underline{E})\phi = i\omega\varepsilon\eta M(\hat{\rho},a)\phi\left(1+\frac{1}{ikr}\right)\underline{i}\times\underline{r_0}$$

$$= ik\,M(\hat{\rho},a)\phi\left(1+\frac{1}{ikr}\right)\left[\sin\theta\sin\varphi\,\underline{k} - \cos\theta\,\underline{j}\right]$$

$(\underline{n}\cdot\underline{H})\,\underline{\nabla\phi} = 0$, Again for $r \ggg 1$, $\left(1+\frac{1}{ikr}\right) \longrightarrow 1$

$$4\pi\,\underline{H}(x) = \int_s ik\phi M(\hat{\rho},a)\left[\underline{j} + \cos\theta\,\underline{j} - \sin\theta\sin\varphi\,\underline{k}\right]da$$

$$H(x) = \frac{ik}{4\pi}\int_s \frac{M(\hat{\rho},a)e^{-ikr}}{r}\left[(1+\cos\theta)\underline{j} - \sin\theta\sin\varphi\,\underline{k}\right]da$$

$$= \frac{2\pi i}{4\pi\lambda}\,\frac{e^{-ikR}}{R}\left[(1+\cos\theta)\underline{j} - \sin\theta\sin\varphi\,\underline{k}\right]\int_s M(\hat{\rho},a)e^{ik\hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)}\hat{\rho}\,d\hat{\rho}\,d$$

$$= \frac{F}{2\pi}\int_s M(\hat{\rho},a)e^{ik\hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)}\hat{\rho}\,d\hat{\rho}\,d\hat{\varphi} \tag{3.15}$$

where $\underline{F} = \frac{i\pi e^{+i(\omega t - kR)}}{\lambda R}\left[(1+\cos\theta)\underline{j} - \sin\theta\sin\varphi\,\underline{k}\right]$ (3.16)

Equation (3.15) is identical in form to equation (3.09) and

$$\underline{H(x)} = \underline{F}\int_0^a M(\hat{\rho}, a)\,\hat{\rho}\,J_0(k\hat{\rho}\sin\theta)\,d\hat{\rho} \tag{3.17}$$

For $M(\hat{\rho}, a) = \bar{M}$; $\underline{H(x)} = \underline{F}\,\bar{M}\,a^2\,\frac{J_1(x)}{x}$ (3.18)

3.8 The solutions for $\underline{E}$(x) and $\underline{H}$(x) as they appear in equations (3.12) and (3.18) are the most useful forms in this analysis but to verify that they are correct in practice one must use the expression for the energy of the distant field(1) since this energy is all that may be measured.

3.9 $$\underline{P} = \tfrac{1}{2}\,\underline{E} \times \underline{\tilde{H}} \tag{3.19}$$

where $\underline{\tilde{H}}$ indicates complex conjugate of $\underline{H}$

$$\underline{\tilde{H}} = -\frac{ik e^{-i(\omega t - kR)}}{R\lambda}\,\bar{M}\left[(1+\cos\theta)\underline{j} - \sin\theta\sin\varphi\,\underline{k}\right] \tag{3.20}$$

$$\underline{P} = -\tfrac{1}{2}\,\frac{(i)^2 k^2}{4R^2\lambda^2}\,\bar{I}\bar{M}\left[a^2\,\frac{J_1(x)}{x}\right]^2\Big[\Big\{(1+\cos\theta)\underline{i} -$$

$$\sin\theta\cos\varphi\,\underline{k}\Big\} \times \Big\{(1+\cos\theta)\underline{j} - \sin\theta\sin\varphi\,\underline{k}\Big\}$$

$$= \tfrac{1}{2}\,\frac{4\pi^2}{\lambda^2}\,\frac{\bar{I}\bar{M}}{4R^2}\left[a^2\,\frac{J_1(x)}{x}\right]^2\Big[(1+\cos\theta)(\sin\theta\ \cos\varphi)\underline{i}$$

$$+ (1+\cos\theta)(\sin\theta\ \sin\varphi)\underline{j} + (1+\cos\theta)^2\underline{k}\Big]$$

$$\underline{P} = \tfrac{1}{2}\,\frac{\pi^2 \eta M^2}{\lambda^2 R^2}\left[a^2\,\frac{J_1(x)}{x}\right]^2\Big[(1+\cos\theta)\Big]\Big[\sin\theta\cos\varphi\,\underline{i} + \sin\theta\sin\varphi\,\underline{j} + \cos\theta\,\underline{k} + \underline{k}\Big]$$

but $\sin\theta\cos\varphi\,\underline{i} + \sin\theta\sin\varphi\,\underline{j} + \cos\theta\,\underline{k} = \underline{r_0}$

and $\underline{k} = \cos\theta\,\underline{r_0} - \sin\theta\,\underline{\theta_0}$

$$\underline{P} = \frac{\eta}{2}\left[\frac{\pi\bar{M}}{\lambda R}\,a^2\,\frac{J_1(x)}{x}\right]^2\Big[1+\cos\theta\Big]\Big[\underline{r_0} + \cos\theta\,\underline{r_0} - \sin\theta\,\underline{\theta_0}\Big]$$

$$= \frac{\eta}{2}\left[\frac{\pi\bar{M}}{\lambda R}\,a^2\,\frac{J_1(x)}{x}\right]^2 (1+\cos\theta)^2\,\underline{r_0} - \frac{\eta}{2}\left[\frac{\pi\bar{M}}{\lambda R}\,a^2\,\frac{J_1(x)}{x}\right]^2 \sin\theta(1+\cos\theta)\,\theta_0$$

In the region where θ is small equation (3.21) reduced to

$$\underline{P} = \frac{\eta}{2}\left[\frac{\pi\bar{M}}{\lambda R}\,a^2\,\frac{J_1(x)}{x}\right]^2 (1+\cos\theta)^2\,\underline{r_0}$$

since $\sin\theta << (1+\cos\theta)^2$ for $\theta < 20°$. This expression for the energy flow agrees with the generally accepted solution given by Schelkunoff (2) for transmission through a circular aperture.

3.10 The remainder of this paper will be confined to relations between the electric field vector distribution and the corresponding aperture distribution. There is no loss in generality in doing this since the energy in distant space is proportional to the square of the amplitude of the electric field vector. Specifically;

$$\underline{E} = \eta\,\underline{H} \quad \text{and} \quad \underline{P} = \tfrac{1}{2}\,\underline{E} \times \widetilde{H} \; ;$$

$$\therefore \; \underline{\Phi} \sim (E)^2$$

Chapter IV

Solution for Aperture Distribution

4.1 Equation (3.11), page . $\underline{E}(x) = \underline{G} \int_{\hat{\rho}=0}^{a} J_0(k\hat{\rho} \sin\theta) I(\hat{\rho},a)\hat{\rho}\, d\hat{\rho}$ is the integral equation that will always arise from a circularly symmetric plane polarized amplitude distribution over an aperture. Its solution for $\underline{E}(x)$ if I $(\hat{\rho},a)$ is known is in general not difficult and may be accomplished by numerical integration if other means fail. However, the solution for I $(\hat{\rho},a)$ when $\underline{E}(x)$ is known presents a different problem.

4.2 One method of attack has been to use Fourier transforms. This requires the assumption that the limits of integration may be extended from 0 to ∞ and considerable error may be introduced since the function I $(\hat{\rho},a)$ may make a large contribution to the integral between a and ∞. If we write:

$$\underline{E}(x) = \underline{G} \int_0^{\infty} J_0(k\hat{\rho}\,\mathrm{Sin}\ \ ) \; I(\hat{\rho}\,,a)\hat{\rho}\; d\hat{\rho}$$

$$-\underline{G}\left\{\int_a^{\infty} J_0(k\hat{\rho}\,\mathrm{Sin}\,\theta)\; I(\hat{\rho}\,,a)\hat{\rho}\; d\hat{\rho}\right\} \qquad (4.01)$$

then the error introduced in solving by Fourier transforms or Fourier - Bessel transforms is equal to the value of the second integral. Obviously, for certain $I(\hat{\rho},a)$ functions this error will be large and only in special cases will it be zero. In practice, it is impossible to have $I(\hat{\rho},a)$ equal to zero at $\hat{\rho} = a$ and hence an error will always be introduced if this method is used. If the unit step function is included as a factor in the aperture distribution and Fourier - Bessel transforms are used, the extension of the limits to infinity introduces no error but this method leads to rather formidable equations except in special cases.

4.3 The method developed here requires no simplifying assumptions regarding limits of integration nor aperture size. The equations are

relatively simple and the integrations are direct.

4.4 Returning to equation (3.11) $\underline{E}(x) = \underline{G}\int_0^a I(\hat{\rho},a)\,\hat{\rho}\,J_0(k\hat{\rho}\,\sin\theta)\,d\hat{\rho}$

Assume that $I(\hat{\rho},a)$ may be written as $I\left[(\hat{\rho}/a)^2\right] = I(Z^2)$ (4.02)

This assumption imposes no further restrictions on $I(\hat{\rho},a)$ since it has already been specified that the aperture distribution be circularly symmetric.

In equation (2.11): let $Z = \hat{\rho}/a$

then $\hat{\rho} = aZ$

$d\hat{\rho} = a\,dZ$

and $k\hat{\rho}\sin\theta = ka\sin\theta\,\frac{\hat{\rho}}{a} = \chi Z$ since $\chi = ka\sin\theta$

The limits of integration become:

for $\hat{\rho} = 0$ $Z = 0$

$\hat{\rho} = a$ $Z = 1$

Hence: $\underline{E}(x) = \underline{G}\int_0^1 I(Z^2)\,aZ\,J_0(\chi Z)\,a\,dZ$

$$= \underline{G}\,a^2\int_0^1 Z\,I(Z^2)\,J_0(\chi Z)\,dZ \tag{4.03}$$

for $I(Z^2) = \sum_{k=0}^{\infty} I_{2k} Z^{2k}$; $ZI(Z^2) = \sum_{k=0}^{\infty} I_{2k} Z^{2k+1}$

$$\text{and } \underline{E(\chi)} = \underline{G}\,a^2\int_0^1 J_0(\chi Z)\sum_{k=0}^{\infty} I_{2k} Z^{2k+1}\,dZ \tag{4.04}$$

$$= \underline{G}\,a^2\sum_{k=0}^{\infty} I_{2k}\int_0^1 J_0(\chi Z)\,Z^{2k+1}\,dZ \tag{4.05}$$

Taking the summation sign outside the integration is permissible here since only distributions which are possible in practice will be considered; to be possible

$Z I(Z^2) = \sum_{k=0}^{\infty} I_{2k} Z^{2k+1}$ must be finite at $Z = 1$, hence $\sum_{k=0}^{\infty} I_{2k}$ must be finite.

4.5 Consider: $\int_0^1 J_0(xZ) Z^{2k+1} dZ = \frac{1}{x^2} \int_0^1 xZ\, J_0(xZ) Z^{2k} d(xZ)$

integrating by parts where $\int \mu \, dv \; \mu v - \int v \, d\mu$

$$\text{let } \mu = Z^{2k}, \quad dv = xZ\, J_0(xZ)\, d(xZ)$$

$$\text{then} \int_0^1 J_0(xZ) Z^{2k+1} dZ = \frac{1}{x^2} \Big\{ xZ J_1(xZ) \cdot Z^{2k} \Big|_{Z=0}^{Z=1}$$

$$-2k \int_0^1 xZ J_1(xZ) Z^{2k-1} dZ = \frac{J_1(x)}{x} - \frac{2k}{x^2} \int_0^1 xZ J_1(xZ) Z^{2k-1} dZ :$$

Continued integration by parts gives

$$\frac{J_1(x)}{x} - \frac{2k}{x^4} \int_0^1 (xZ)^2 J_1(xZ) Z^{2k-2} dZ = \frac{J_1(x)}{x} -$$

$$\frac{2k}{x^4} \Big\{ (xZ)^2 J_2(xZ) Z^{2k-2} \Big|_{Z=0}^{Z=1} - (2k-2) \int_0^1 (xZ)^2 J_2(xZ) Z^{2k-3} dZ$$

$$= \frac{J_1(x)}{x} - 2k \frac{J_2(x)}{x^2} + \frac{2k(2k-2)}{x^4} \int_0^1 (xZ)^2 J_2(xZ) Z^{2k-3} dZ$$

Further integration gives the series:

$$\frac{J_1(x)}{x} - 2k \frac{J_2(x)}{x^2} - \frac{2k(2k-2)}{x^3} J_3(x) - \frac{2k(2k-2)(2k-4)}{x^4} J_4(x) + \ldots.$$

$$= \tfrac{1}{2} \Big[\frac{J_1(x)}{x/2} - k \frac{J_2(x)}{(x/2)^2} + k(k-1) \frac{J_3(x)}{(x/2)^3} - k(k-1)(k-2) \frac{J_4(x)}{(x/2)^4} + \ldots.$$

$$= \tfrac{1}{2} \Big[\frac{\Lambda_1(x)}{1!} - k \frac{\Lambda_2(x)}{2!} + k(k-1) \frac{\Lambda_3(x)}{3!} - k(k-1)(k-2) \frac{\Lambda 4(x)}{4!} +$$

Using this result we may write equation (4.05) as

$$\underline{E(x)} = \frac{Ga^2}{2}\left\{ I_0 \Lambda_1(x) + \sum_{k=0}^{\infty} \sum_{j=1}^{k+1} (-1)^{j+1} I_{2k} \left[\frac{k(k-1)(k-2)\cdots(k-j+2)\Lambda_j(x)}{j!} \right] \right\} \tag{4.06}$$

or

$$\underline{E(x)} = \frac{Ga^2}{2}\left\{ \Lambda_1(x)\left[I_0 + I_2 + I_4 + I_6 + \ldots\ldots\ldots\ldots \right] \right.$$

$$-\frac{\Lambda_2(x)}{2!}\left[0 + I_2 + 2I_4 + 3I_6 + 4I_8 + 5I_{10} + \ldots\ldots \right]$$

$$+\frac{\Lambda_3(x)}{3!}\left[0 + 0 + 2I_4 + 3.2I_6 + 4.3I_8 + 5.4I_{10} + \ldots\ldots \right]$$

$$-\frac{\Lambda_4(x)}{4!}\left[0 + 0 + 0 + 3.2I_6 + 4\cdot3\cdot2I_8 + 5\cdot4\cdot3\cdot I_{10} + \ldots \right] \cdots\cdots$$

$$(-1)^{p+1}\frac{\Lambda_p(x)}{p!}\left[(p-1)!\,I_{2p-2} + \frac{p!}{1!} I_{2p} + \frac{(p+1)!}{2!} I_{2p+2} \right.$$

$$+ \frac{(p+2)!}{3!} I_{2p+4} + \ldots\ldots$$

or

$$\underline{E(x)} = \frac{Ga^2}{2}\sum_{p=1}^{\infty} (-1)^{p+1} \frac{\Lambda_p(x)}{p!} \sum_{k=0}^{\infty} \frac{(p-1+k)!}{k!} I_{(2p-2+2k} \tag{4.07}$$

4.6 Now: If the desired pattern is expressed in terms of $\Lambda_p(x)$ and if we consider the $\Lambda_p(x)$ as the independent variable we can equate coefficients of the $\Lambda_p(x)$ and solve for the I_{2k}. The requirement that the desired pattern be expressed in terms of $\Lambda_p(x)$ is not a restriction imposed by the method so much as a restriction imposed by nature. The reason for this is discussed in section 5.3 page .

4.7 The following two examples solved for distant space patterns for which the corresponding aperture distributions are known show in detail

the method of solving for $I(Z^2)$.

Example 1: Let $\underline{E}(x) = \underline{G}\ \bar{I}\ a^2\ \frac{J_1(x)}{x}$ for which $I(Z^2) = \bar{I}$ (2) that is constant amplitude.

$\underline{E}(x) = \underline{G}\bar{I}a^2\frac{J_1(x)}{x} = \frac{1}{2}\underline{G}\bar{I}a^2\Lambda_1(x).$ Equating coefficients:

$$\frac{1}{2}\underline{G}\bar{I}a^2 = \frac{\underline{G}a^2}{2}\Big[I_0 + I_2 + I_4 + I_6 + \ldots\ldots$$

$$0 = -\frac{\underline{G}a^2}{2\cdot 2!}\Big[0 + I_2 + 2I_4 + 3I_6 + \ldots\ldots$$

$$0 = \frac{\underline{G}a^2}{2\cdot 3!}\Big[0 + 0 + 2I_4 + 3\cdot 2I_6 + \ldots\ldots$$

$$0 = \ldots\ldots$$

$$0 = (-1)^{n-1}\frac{\underline{G}a^2}{2\cdot n!}\Big[0+0+0+0+(n-1)!\,I_{2n-2} + \ldots\ldots$$

$$0 = \ldots\ldots$$

From the rules for evaluating determinants it is apparent that $I_{2k} = 0$ for $k \geq 1$

$$\therefore\ \frac{1}{2}\underline{G}\bar{I}a^2 = \frac{\underline{G}a^2}{2}\,I_0$$

$I_0 = \bar{I}$; $I(Z^2) = \bar{I}$ and

$\underline{I}(\hat{\rho}, a) = \bar{I}\,\underline{i}$ which is known to be correct

Example 2: Let $\underline{E}(x) = 2\bar{I}\underline{G}a^4\frac{J_2(x)}{x^2} = \frac{\bar{I}\underline{G}a^4}{4}\Lambda_2(x)$

for which $I(Z^2) = \bar{I}a^2(1 - Z^2)$ ***

Equating coefficients:

$$0 = \frac{\underline{G}a^2}{2}\Big[I_0 + I_2 + I_4 + I_6 + \ldots\ldots$$

$$\frac{\bar{I}\underline{G}a^4}{4} = -\frac{\underline{G}a^2}{2\cdot 2!}\Big[0 + I_2 + 2I_4 + 3I_6 + \ldots\ldots$$

$$0 = \frac{\underline{G}a^2}{2\cdot 3!}\Big[0 + 0 + 2I_4 + 3\cdot 2I_6 + \ldots\ldots$$

** Appendix IV

*** This may be readily shown using equation (3.11) p. .

Again $I_{2k} \equiv 0$ for $\geqslant 2$ and we have:

$$\left.\begin{aligned} 0 &= I_0 + I_2 \\ -\bar{I}a^2 &= 0 + I_2 \end{aligned}\right\} \begin{aligned} I_2 &= -\bar{I}a^2 \\ I_0 &= \bar{I}a^2 \end{aligned}$$

$$I(z^2) = \bar{I}a^2(1 - z^2) \text{ and}$$

$$\underline{I(\hat{\rho}, a)} = \bar{I}a^2(\underline{I} - (\hat{\rho}/a)^2]\ \underline{i}$$

4.9 For a general solution assume that $\underline{E}(x) = \bar{I}\,\underline{G}\,\Lambda_m(x)$ and equate coefficients of $\Lambda_m(x)$ in equation (4.07)

$$\bar{I}\,\underline{G} = \frac{\underline{G}a^2}{2}\,\frac{(-1)^{m+1}}{m!}\sum_{k=0}^{\infty}\frac{(m-1+k)!}{k!}\,I_{(2m-2+2k)}$$

but

$$I_{(2m-2+k)} \equiv 0 \text{ for } k > 0^{**} \text{ and}$$

$$\bar{I} = \frac{a^2}{2}\,\frac{(-1)^{m+1}}{m!}\,(m-1)!\ I_{(2m-2)}$$

$$= \frac{a^2}{2}\,\frac{(-1)^{m+1}}{m}\,I_{(2m-2)}$$

$$\therefore I_{(2m-2)} = \frac{2m\bar{I}}{a^2}\,(-1)^{m+1} \tag{4.08}$$

However, I_{2k} for $2k < 2m-2$ are not equal to zero[**]. If we equate coefficients of $\Lambda_{(m-j)}$ ($j = 1, 2, 3, \ldots, m-1$) in equation (4.07) for $\underline{E}(x) = \bar{I}\,\underline{G}\,\Lambda_m(x)$; then $p = m-j$ in equation (4.07) and we have

$$0 = \frac{\underline{G}a^2}{2}\,\frac{(-1)^{m-j+1}}{(m-j)!}\sum_{k=0}^{\infty}\frac{(m-j-1+k)!}{k!}\,I_{(2m-2j-2+2k)} \quad \text{or}$$

$$\sum_{k=0}^{\infty}\frac{(m-j-1+k)!}{k!}\,I_{(2m-2j-2+2k)} = 0 \tag{4.09}$$

$$j = 1, 2, 3, \ldots, m-1$$

[**] Appendix IV

Equation (4.09) provides a ready means of solving for the I_{2k} for $2k < 2m-2$.

For $j = 1$ $$\frac{(m-2)!}{0!} I_{(2m-4)} + \frac{(m-1)!}{1!} I_{(2m-2)} = 0$$

Substituting from equation (4.08) for $I_{(2m-2)}$ in the above expression,

$$I_{(2m-4)} = -\frac{2\bar{I}}{a^2}(-1)^{m+1}\frac{m(m-1)}{1!} \qquad (4.10)$$

For $j = 2$ we may solve for $I_{(2m-6)}$ and

$$I_{(2m-6)} = \frac{2\bar{I}}{a^2}(-1)^{m+1} m\frac{(m-1)(m-2)}{2!} \qquad (4.11)$$

The general expression for I_{2k}, $2k < 2m-2$, is

$$I_{[2m-2(j+1)]} = (-1)^j \frac{2\bar{I}}{a^2}(-1)^{m+1} m\frac{(m-1)(m-2)\cdots\cdots(m-j)}{j!} \qquad (4.12)$$

and $I(Z^2)$ for $\underline{E}(x) = I\,\underline{G}\,\Lambda_m(x)$ is

$$I(Z^2) = \sum_{j=0}^{m=1} I_{[2m-2(j+1)]} Z^{[2m-2(j+1)]} \qquad (4.13)$$

$$= \frac{2\bar{I}}{a^2}(-1)^{m+1}\sum_{j=0}^{m-1}(-1)^j \frac{m(m-1)(m-2)\cdots\cdots(m-j)}{j!} Z^{2m-2(j+1)} \qquad (4.14)$$

If (4.14) is rearranged so that low powers of Z appear in the first of the summation then

$$I(Z^2) = \frac{2\bar{I}}{a^2}(-1)^{m+1}\left\{(-1)^{m-1}\frac{m!}{(m-1)!}\frac{Z^0}{0!} + (-1)^{m-2}\frac{m!}{(m-2)!}\frac{Z^2}{1!}\right.$$
$$\left. + (-1)^{m-3}\frac{m!}{(m-3)!}\frac{Z^4}{2!} + \cdots\cdots\right\}$$
$$= \frac{2\bar{I}}{a^2}\left\{\frac{mZ^0}{0!} - m\frac{(m-1)Z^2}{1!} + m\frac{(m-1)(m-2)Z^4}{2!} + \cdots\cdots\right\}$$
$$= \frac{2\bar{I}}{a^2}\sum_{j=0}^{m-1} m\frac{(m-1)(m-2)\cdots\cdots(m-j)}{j!} Z^{2j} \qquad (4.15)$$

which is readily recognized as the binomial expansion of $\frac{2\bar{I}}{a^2} m(1-Z^2)^{m-1}$

Therefore, the required aperture distribution to give a distant space

electric field vector distribution of $\underline{E}(x) = I\, \underline{G}\, \Lambda_m(x)$ is

$$I(Z^2) = \frac{2\bar{I}}{a^2}\, m(1-Z^2)^{m-1} \qquad \text{or}$$

$$I(Z^2) \sim m(1-Z^2)^{m-1} \tag{4.16}$$

4.10 Several interesting facts may be deduced from equation (4.16). Only when $\Lambda_1(x)$ is present in this distribution in distant space will the aperture distribution have a value at the edge of the aperture where $Z = 1$. Since in practice there will always be some energy at the edge of the aperture we may conclude that $\underline{E}(x)$ will always have $\Lambda_1(x)$ in it. This is important since it tends to introduce side lobes at low values of x. See Fig. 3, Appendix III. To radiate a lobeless pattern this tendency of $\underline{E}(x)$ to go to zero at the zeros of $\Lambda_1(x)$ and further, to go negative (i.e. phase reversal) between alternate zeros of $\Lambda_1(x)$ must be overcome by higher $\Lambda(x)$ functions. But the introduction of higher $\Lambda(x)$ functions tends to broaden the pattern. This forces a compromise between beam width characteristics and sidelobe characteristics. The general tendency, as has been known from experiment, is that to radiate a narrow pattern means the introduction of sidelobes and vice versa, the reduction or elimination of sidelobes tends to broaden the pattern. Theoretically, from an examination of equation (4.16), it is seen that a lobeless pattern may be radiated without infinite energies over the aperature. All that is necessary is to choose $\Lambda_m(x)$ such that the first zero of $\Lambda_m(x)$ occurs at $x = ka \sin\theta = ka$ (i.e. at $\theta = 90°$). Note here, that since $k = \frac{2\pi}{\lambda}$, and that λ is inversely proportional to the frequency the value of x increases as the frequency of the radiated energy increases and further, the order of $\Lambda_m(x)$ must be increased. Since $I(Z^2) \sim m(1-Z^2)^{m-1}$, the energy at the center of the aperture must be more and more sharply peaked to radiate a lobeless pattern as the frequency of the radiated energy is increased.

The same is true as the dimensions of the aperture are increased. Fig. 4, Appendix III, shows the required aperture distribution to give a lobeless pattern for a frequency of radiation and aperture size such that ka = 10. Theoretically, a distant space distribution of Λ_8 (x) will give the desired lobeless pattern, however, in order that the practical requirement that the energy not be equal to zero at the edge of the aperture be met, a distribution of $\Lambda_1(x)+\Lambda_8(x)$ has been chosen. Note the shoulders that appear in this pattern at $\theta = 50°$ and $\theta = 90°$. These can be reduced only by increasing the proportion of $\Lambda_8(x)$ over the amount of Λ_1 (x). Increasing the proportion of Λ_8 (x) requires a reduction of the relative amount of energy at the edge of the aperture compared to the amount of energy at the center. In the limit, complete elimination of these shoulders may be obtained by reducing the energy at the edge of the aperture to zero and radiating a pure $\Lambda_8(x)$ pattern.

4.11 Figs. 5 and 5a, Appendix III, show clearly the effect on the required aperture distribution of narrowing the radiated pattern by choosing a distant space distribution containing a higher order Λ (x) function which is subtractive. In Fig. 5, showing the distant field, a comparison is made between a Λ_6 pattern and a $2\Lambda_6 - \Lambda_{10}$ pattern. The half power point of the former occurs at X = 3.1 and that of the latter occurs at X = 2.6. A reduction of some 16% is obtained. Fig. 5a shows the comparison of aperture distributions corresponding to the two above fields. It will be seen that the distribution corresponding to the narrower pattern is hollow in the center of the aperture. Such distributions are always observed in center fed paraboloids and are caused by the feed support. The hollowing out of energy at the center has narrowed the distant space pattern, at the same time it has increased the amount of energy in the sidelobes. It appears, however, that in theory at least, a postulated field pattern could

be approximated as closely as desired by a Fourier-Bessel series of Λ functions and then the required aperture distribution could be readily obtained using equation (4.16) page 26. An attempt at such an analysis is beyond the scope of this paper due to time limitations.

4.12 Two additional postulated distant space distributions with their corresponding aperture distributions are plotted in Fig. 6 and 7. In Fig. 6 curve 1 is the distant space distribution for

$$\Phi \backsim (\Lambda_1 + 2\Lambda_3 + \Lambda_5 + \Lambda_7 - \Lambda_2 - \Lambda_4 - \Lambda_6)^2$$

It shows the result that may be obtained from a combination of Λ functions. The resultant pattern is lobeless to ka sin θ = 10. Its half power point occurs at X = 4.2. Curve 2 of this same Figure shows the effect of subtracting Λ_{10} from the distant space electric field distribution which gave rise to curve 1.. The half power point now occurs at X = 3.2, a reduction of nearly 24%. Note the large increase of energy in the first side lobe which begins at X = 3.6. The pattern has been narrowed but at the expense of poor sidelobe structure. Fig. 6a shows the aperture distribution corresponding with curve 1 of Fig. 6; Fig. 6b shows the aperture distribution corresponding with curve 2 of Fig. 6.

Figure 7 is the aperture distribution for a pattern having no center lobe. Fig. 7a is the aperture distribution corresponding with this space pattern.

4.13 In addition to these pattern, Fig. 3 is a plot of the Λ functions for order 1 through 12 plus 16 and 20. A table of values for all Λ_p functions, 1 through 20, for values of argument zero to 10 is tabulated on the pages following Fig. 3. These values have been taken from "Table of Spherical Bessel Functions", Volume II, 1947, prepared by the Mathematical Tables Project, National Bureau of Standards. The page following contains powers of $(1 - Z^2)$ evaluated at intervals of one tenth for $0 \lesssim Z \lesssim 1.0$ and is included for ready reference.

Chapter V

Theoretically Possible Patterns

5.1 Starting again with equation (4.05) page :

$$\underline{E}(x) = \underline{G}a^2 \sum_{k=0}^{\infty} I_{2k} \int_0^1 J_0(xz)\, z^{2k+1}\, dz \qquad (4.05)$$

Since

$$J_0(xz) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!n!} \left(\frac{x}{2}\right)^{2n} z^{2n} \qquad (7)$$

$$\underline{E}(x) = \underline{G}a^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{2k}(-1)^n}{n!n!} (x/2)^{2n} \int_0^1 z^{2k+2n+1}\, dz$$

$$= \underline{G}a^2 \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{2k}(-1)^n}{n!n!} (x/2)^{2n} \left. \frac{z^{2k+2n+2}}{2k!2n+2} \right|_{z=0}^{z=1}$$

$$= \frac{\underline{G}a^2}{2} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{I_{2k}(-1)^n}{n!n!(k+n+1)} (x/2)^{2n} \qquad (5.01)$$

If now $\underline{E}(x)$ is expressed in series form in powers of $(x/2)^{2n}$ $n = 0,1,2,3..$ we may equate coefficients of $(x/2)^{2n}$ and solve for the I_{2k}.

5.2 This method permits a theoretical solution whenever the field distribution may be expressed as a power series and is more general than the method given in Chapter II. However, the real worth of this method is that it provides a rather easy means of investigating a proposed pattern to determine if it is theoretically possible. This may be shown as follows.

5.3 Let $\underline{G}\, a^2 A$ be the coefficient of $(x/2)^{2n}$ of the proposed pattern, then equating coefficients from equation (5.01)

$$A = \tfrac{1}{2} \sum_{k=0}^{\infty} I_{2k} \frac{(-1)^m}{m!m!} \frac{1}{(k+m+1)}$$

$$\frac{2\, m!m!}{(-1)^m} A = \sum_{k=0}^{\infty} \frac{I_{2k}}{k+m+1} = \frac{1}{m+1} \sum_{k=0}^{\infty} \frac{I_{2k}}{1+k/m+1} \qquad \text{or}$$

$$2(m+1)!\, m!\, (-1)^m A = \sum_{k=0}^{\infty} \frac{I_{2k}}{1+k/m+1} \qquad (5.02)$$

If we consider a finite number of I_{2k}, and the I_{2k} are bounded, which they must be for the aperture distribution to be possible in practice, then as m increases towards infinity we may write equation (5.02) as

$$\underset{m \to \infty}{2(m+1)!\,(m)!\,(-1)^{m} A} = \underset{m \to \infty}{\sum_{k=0}^{B} I_{2k}} \text{ where } B \text{ is finite} \qquad (5.03)$$

In this expression for the I_{2k} their sum will remain finite only if A is of such form that it cancels the $(m+1)!\, m!\, (-1)^{m}$ on the left. This requires that $\underline{E}(x)$ be a sum of Bessel Functions or similar functions, or combination thereof, and only in special cases will the distant space distribution not be periodic.

5.4 The requirement that $\underline{E}(x)$ be expressible in terms of Bessel Functions is not in itself sufficient to establish that such a pattern is theoretically possible. Only a sum of single Bessel or similar functions will satisfy the requirements that the coefficients of A in equation (5.03) cancel the factorial expression $(m+1)!\, m!\, (-1)^{m}$

5.6 As an example consider the field vector pattern

$$\underline{E(x)} = \bar{I}\,\underline{G}\,a^{2}\, \frac{J_1(x)}{x}\, e^{-\left(\frac{\alpha x}{2}\right)^{2}} \qquad (5.04)$$

Such a pattern would permit the reduction of side lobes to as small a value as desired if such a pattern were possible. It is known to be impossible in practice and it may be shown to be impossible in theory by applying the method given above. To do this

let
$$\frac{J_1(x)}{x} = \sum_{m=0}^{\infty} \frac{(-1)^{m}}{2\,m!\,(m+1)!} \left(x/2\right)^{2m}$$

$$e^{-\left(\frac{\alpha x}{2}\right)^{2}} = \sum_{j=0}^{\infty} \frac{(-1)^{j} \alpha^{2j}}{j!} \left(x/2\right)^{2j}$$

$$\underline{E}(x) = \frac{\bar{I}\,\underline{G}\,a^{2}}{2} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j} \alpha^{2j}}{j!\, m!\, (m+1)!} \left(x/2\right)^{2(m+j)} \qquad (5.05)$$

let $m+j=n$ and equate coefficients of $(x/2)^{2n}$

$$\frac{\bar{I}\underline{G}a^2}{2}\,\frac{(-1)^n\alpha^{2j}}{j!\,m!\,(m+1)!} = \frac{Ga^2(-1)^n}{n!\,n!}\sum_{k=0}^{\infty}\frac{I_{2k}}{2k+2n+2}$$

or for n very large $\bar{I}\,\frac{\alpha^{2j}(n+1)\,n!}{j!\,m!\,(m+1)!} = \sum_{k=0}^{\infty} I_{2k}$

let $m=0$ then j = n and

$I\alpha^{2n}(n+1)! = \sum_{k=0}^{\infty} I_{2k}$ and the $I_{2k} \longrightarrow \infty$ as n $\longrightarrow \infty$

This is obvious for $\alpha > 1$. For $\alpha < 1$ let $\alpha = 1/\beta$, $\beta > 1$ and consider the ratio of the n^{th} and $(n + 1)^{th}$ terms. We have n^{th} term $= \bar{I}\,\frac{(n+1)!}{\beta^{2n}}$ and the $(n + 1)^{th}$ terms $= \bar{I}\,\frac{(n+2)!}{\beta^{2(n+1)}}$

Their ratio will be $\bar{I}\,\frac{(n+2)!}{\beta^{2(n+1)}}\;\frac{\beta^{2n}}{\bar{I}\,(n+1)!} = \frac{n+2}{\beta^2}$

and as n $\longrightarrow \infty$ this ratio becomes infinitely large, and the $I_{2k} \longrightarrow \infty$.

Hence, we conclude that the pattern $\underline{E}(x) = \bar{I}\underline{G}a^2\,\frac{J_1(x)}{x}\,e^{-\left(\frac{\alpha x}{2}\right)^2}$ is impossible in theory as well as in practice.

5.6 In a similar manner it may be shown that products of Bessel Functions, such as $\frac{J_1(x)}{x}\,\frac{J_2(x)}{x^2}$ are theoretically impossible.

Again use $\frac{J_1(x)}{x} = \sum_{m=0}^{\infty}\frac{(-1)^m}{2m!\,(m+1)!}\left(x/2\right)^{2m}$

and $$\frac{J_2(x)}{x^2} = \sum_{j=0}^{\infty}\frac{(-1)^j}{4j!\,(j+2)!}\left(x/2\right)^{2j} \qquad (5.06)$$

then $$\frac{J_1(x)}{x}\,\frac{J_2(x)}{x^2} = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+j}\,\left(x/2\right)^{2(m+j)}}{8m!\,j!\,(m+1)!\,(j+2)!} \qquad (5.07)$$

and for m + j = n we may equate the coefficients of $(x/2)^{2n}$.

This gives

$$\frac{\bar{I}\underline{G}a^2}{2}\,\frac{(-1)^n}{8m!\,j!\,(m+1)!\,(j+2)!} = \frac{\underline{G}a^2(-1)^n}{n!\,n!}\sum_{k=0}^{\infty}\frac{I_{2k}}{2k+2n+2m}$$

or $$\frac{\overline{I}}{8}\,\frac{(n+1)!\,n!}{m!\,j!\,(m+1)!\,(j+2)!} = \sum_{k=0}^{\infty} \frac{I_{2k}}{1+k/n+1} \tag{5.08}$$

Let $j = m$, then $n = 2m$ and

$$\lim_{m\to\infty} \frac{\overline{I}}{8}\,\frac{(2m+1)!\,(2m)!}{m!\,m!\,(m+1)!\,(m+2)!} = \lim_{m\to\infty} \sum_{k=0}^{\infty} I_{2k} \tag{5.09}$$

An inspection of the ratio of n^{th} and the $(n + 1)^{th}$ term shows that the $I_{2k} \to \infty$ since

$$\frac{(n+1)^{th}}{n^{th}} = \lim_{m\to\infty} \frac{(2m+2)!\,(2m+1)!}{m!\,(m+1)!\,(m+1)!\,(m+3)!}\;\frac{m!\,m!\,(m+1)!\,(m+2)!}{(2m+1)!\,(2m)!}$$

$$= \lim_{m\to\infty} \frac{(2m+2)(2m+1)}{(m+1)\,(m+3)} = \lim_{m\to\infty} \frac{(2+2/m)(2+1/m)}{(1+1/m)(1+3/m)} = 4$$

and we conclude that the pattern $\underline{E}(x) \sim \frac{J_1(x)}{x}\;\frac{J_2(x)}{x^2}$ is impossible.

Chapter VI

Solution for Aperture Distributions Which are Independent of Radius

6.1 A method of solving for $I(\hat{\rho}, a)$ in equation (3.11) page when $I(\hat{\rho}, a) = I(\hat{\rho})$, that is, I is not a function of "a" may be readily obtained as follows.

$$\underline{E}(x) = \underline{E}(ka\sin\theta) = \underline{G}\int_0^a I(\hat{\rho}) J_0(k\hat{\rho}\sin\theta)\hat{\rho}\, d\hat{\rho} \tag{6.01}$$

$$\frac{d}{da}\underline{E}(x) = \underline{G}\left\{\int_0^a \frac{d}{da}\left[I(\hat{\rho})J_0(k\hat{\rho}\sin\theta)\hat{\rho}\right]d\hat{\rho} + I(a)J_0(ka\sin\theta)a\right\} \tag{6.02}$$

Since the integrand of the integral is independent of "a" its derivative will be zero and hence the value of the integral will also be zero and

$$\frac{d}{da}\underline{E}(x) = \underline{G}\, a\, I(a) J_0(ka\sin\theta)$$

Hence, $$I(a) = \frac{\frac{d}{da}E(ka\sin\theta)}{\underline{G}a\, J_0(ka\sin\theta)} \tag{6.03}$$

Example I: for $\underline{E}(x) = G\bar{I}a^2\,\frac{J_1(x)}{x}$ we have

$$I(a) = \frac{G\bar{I}\frac{d}{da}a^2\,\frac{J_1(ka\sin\theta)}{ka\sin\theta}}{\underline{G}\, a\, J_0(ka\sin\theta)} = \frac{\bar{I}\, a J_0(ka\sin\theta)}{a J_0(ka\sin\theta)} = \bar{I}$$

Hence,

$$I(\hat{\rho}) = \bar{I} \quad \text{and} \quad \underline{I}(\hat{\rho}, a) = \bar{I}\,\underline{i}$$

Example 2: for $\underline{E}(x) = \bar{I}\underline{G}a^4\left(\frac{J_1(x)}{x} - \frac{2J_2(x)}{x^2}\right)$ we have

$$\frac{d}{da}\underline{E}(ka\sin\theta) = \frac{d}{da}\bar{I}\underline{G}\left[a^2\frac{(ka\sin\theta)J_1(ka\sin\theta)}{(k\sin\theta)^2} - \frac{2}{(k\sin\theta)^4}(ka\sin\theta)^2 J_2(ka\sin\theta)\right]$$

$$= \bar{I}\underline{G}\left[a^2\frac{(ka\sin\theta)J_0(ka\sin\theta)(k\sin\theta)}{(k\sin\theta)^2} + 2\frac{a^2 J_1(ka\sin\theta)}{k\sin\theta}\right.$$

$$\left. - \frac{2(ka\sin\theta)^2}{(k\sin\theta)^4} J_1(ka\sin\theta)\, k\sin\theta\right]$$

$$= \bar{I}\,\underline{G}\left[a^3 J_0(ka \sin\theta)\right]$$

and

$$I(a) = \frac{\bar{I}\,\underline{G}\, a^3 J_0(ka \sin\theta)}{\underline{G}\, a\, J_0(ka \sin\theta)} = a^2$$

which may be readily verified by direct integration of equation (6.01) for $I(\hat{\rho}) = \hat{\rho}^2$.

6.2 While this method is quite limited in practice it is easy to check a possible solution this way and if the expression for I(a) in equation (6.03) results in a solution for I which is independent of θ the problem is readily solved. The space patterns which will give such a solution are those which arise from integration of equation (6.01).

Chapter VII

Conclusions

7.1 By following the accepted field equations of Maxwell and retaining actual limits of integration a method has been developed permitting correlation of the distant field with a circularly symmetric aperture distribution. This is in distinction to earlier methods based on Fourier transforms in which the limits of integration were extended to infinity, and the results of which were not in general realistic. The method developed here relies on the Λ functions and gives realistic expressions for the lobe structure of distant space patterns. Another valuable result of this work is the definition of the types of pattern that are physically possible, consistent with electromagnetic theory.

7.2 The method developed gives a close insight into the dependence of beam width, and therefore gain of the pattern, on the tolerance of lobe levels. As the pattern functions are made to converge to lobeless patterns the corresponding aperture distributions are found to converge toward a point source centrally located on the aperture. The relatively narrow and low lobe patterns obtained in better radar equipment are found to correspond to certain types of hollow aperture distributions which correspond closely to those obtained in this analysis.

7.3 All possible circularly symmetric amplitude distributions over an aperture set up distant field distributions that are expressible as sums of Λ functions. The absence of a Fourier - Bessel Λ Function method of developing pattern expressions is a distinct limitation to this approach to the problem.

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Appendix I

Table of Symbols (MKS units)

Symbol	Definition
a	radius of aperture
area 1	area within aperture
area 2	area outside aperture in plane of aperture
c	velocity of light $c = \frac{1}{\sqrt{\mu\varepsilon}}$
C	contour bounding aperture
da	differential element of area 1. $da = \hat{\rho}\, d\hat{\rho}\, d\hat{\varphi}$
$\underline{ds}$	vector differential element of length about contour C.
$\underline{E(x',y'z')}$	electric vector within volume V.
$\underline{E}(x)$	electric field vector distribution in distant space
$\underline{F}$	directrix, time and range factor of $\underline{H(x)}$
f	frequency
$\underline{G}$	directrix, time and range factor of $\underline{E}(x)$
$\underline{H(x',y',z')}$	magnetic vector within volume V
$\underline{H(x)}$	magnetic field vector distribution in distant space
i	square root of minus 1
$\underline{i}$	unit vector in $\hat{x}$ direction
$\underline{I(\hat{\rho}\ ,a)}$	electric field vector distribution over aperture
$I(z^2)$	amplitude distribution of electric field vector over aperture
I	amplitude of $I(\hat{\rho}\ ,a)$ for constant amplitude distribution
$\underline{j}$	unit vector in $\hat{y}$ direction
J_m	Bessel function of first kind, integral order m
$\underline{J}$, $\underline{J}^*$	current density
k	phase constant
$\underline{k}$	unit vector in z direction

Symbol	
$\underline{K}, \underline{K}^*$	electric and magnetic current surface densities
$\underline{M}(\hat{\rho}, a)$	magnetic field vector distribution over aperture
M	amplitude of $M(\hat{\rho}, a)$ for constant amplitude distribution
$M(z^2)$	amplitude distribution of magnetic field vector over aperture
$\underline{n}$	unit vector perpendicular to aperture in direction of radiation
$R, \theta, \varphi,$	space polar coordinate system $\underline{r}_o, \underline{\theta}_o, \underline{\varphi}_o$ - unit vectors
$\underline{R}$	position vector from center of aperture to point (R, θ, φ)
$\underline{r}$	position vector from point $(\hat{x}, \hat{y})$ in aperture to point (R, θ, φ)
$\underline{r}_o$	unit vector in R direction
x	$x \equiv ka \sin\theta$
$\hat{x}, \hat{y}, \hat{z}$	rectangular coordinate system $\underline{i}, \underline{j}, \underline{k}$ - unit vectors
z	$z \equiv \hat{\rho}/a$
β	angle between $\underline{R}$ and $\hat{\rho}$
ε	permittivity of space
η	impedance of space $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ for conductance 0
$\underline{\theta}_o$	unit vector in θ direction
λ	free space wavelength
Λ_p	Lambda function of integral order p
μ	permeability of space
ρ, ρ^*	charge density
$\hat{\rho}$	radius vector in aperture
σ, σ^*	line densities of electric and magnetic charge
$\hat{\varphi}$	angular measure in aperture
$\underline{\varphi}_o$	unit vector in φ direction

Symbol	Definition
ϕ	$\phi \equiv \frac{e^{-ikr}}{r}$
Φ	energy at point in distant field
ψ	angle between $\underline{R}$ and $\underline{i}$
ω	angular velocity $\omega = 2\pi f$

Appendix II

Vector Identities

(1) $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$

(2) $\nabla\underline{(\phi + \psi)} = \underline{\nabla\phi} + \underline{\nabla\psi}$

(3) $\nabla\underline{(\phi\psi)} = \phi\,\underline{\nabla\psi} + \psi\,\underline{\nabla\phi}$

(4) $\underline{\nabla}\cdot(\underline{a}+\underline{b}) = \underline{\nabla}\cdot\underline{a} + \underline{\nabla}\cdot\underline{b}$

(5) $\underline{\nabla}\times(\underline{a}+\underline{b}) = \underline{\nabla}\times\underline{a} + \underline{\nabla}\times\underline{b}$

(6) $\underline{\nabla}\cdot(\phi\underline{a}) = \underline{a}\cdot\underline{\nabla\phi} + \phi\,\underline{\nabla}\cdot\underline{a}$

(7) $\underline{\nabla}\times(\phi\underline{a}) = \underline{\nabla\phi}\times\underline{a} + \phi\,\underline{\nabla}\times\underline{a}$

(8) $\underline{\nabla}(\underline{a}\cdot\underline{b}) = (\underline{a}\cdot\underline{\nabla})\underline{b} + (b\cdot\nabla)\underline{a} + \underline{a}\times(\underline{\nabla}\times\underline{b}) + \underline{b}\times(\underline{\nabla}\times\underline{a})$

(9) $\underline{\nabla}\cdot(\underline{a}\times\underline{b}) = \underline{b}\cdot\underline{\nabla}\times\underline{a} - \underline{a}\cdot\underline{\nabla}\times\underline{b}$

(10) $\underline{\nabla}\times(\underline{a}\times\underline{b}) = \underline{a}\,\underline{\nabla}\cdot\underline{b} - \underline{b}\,\underline{\nabla}\cdot\underline{a} + (\underline{b}\cdot\underline{\nabla})\underline{a} - (\underline{a}\cdot\underline{\nabla})\underline{b}$

(11) $\underline{\nabla}\times\underline{\nabla}\times\underline{a} = \underline{\nabla}\,\underline{\nabla}\cdot\underline{a} - \nabla^2\underline{a}$

(12) $\underline{\nabla}\times\underline{\nabla\phi} \equiv 0$

(13) $\underline{\nabla}\cdot\underline{\nabla}\times\underline{a} \equiv 0$

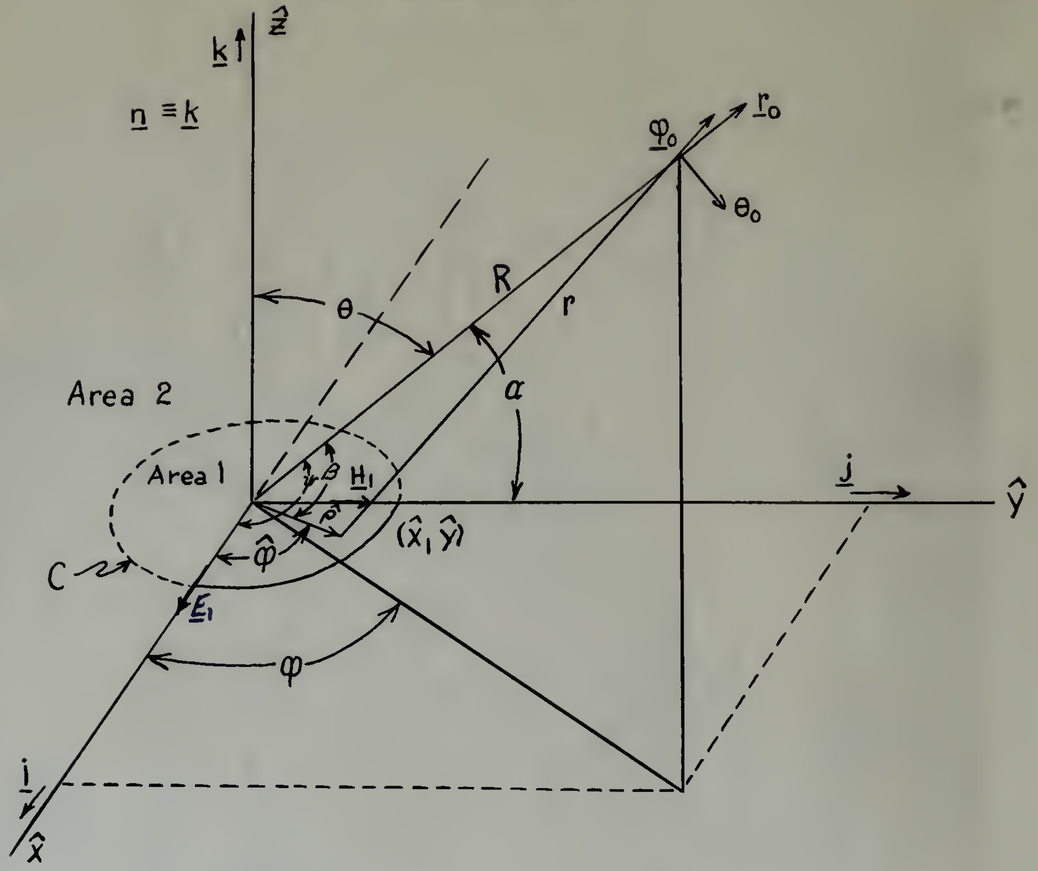

Figure 2

From Fig. 2

$$\frac{R_{\hat{z}}}{R} = \cos\theta \; ; \quad \frac{R_{\hat{y}}}{R} = \cos\alpha \; ; \quad \frac{R_{\hat{x}}}{R} = \cos\psi$$

$$R_{\hat{y}} = R\cos\alpha = R\sin\theta\sin\varphi$$

$$R_{\hat{x}} = R\cos\psi = R\sin\theta\cos\varphi \; ; \quad \cos\psi = \sin\theta\cos\varphi$$

$$R\cos\beta = R\sin\theta\cos(\hat{\varphi}-\varphi) \quad \therefore \cos\beta = \sin\theta\cos(\hat{\varphi}-\varphi)$$

$$r = \sqrt{R^2 + \hat{\rho}^2 - 2R\hat{\rho}\cos\beta} \quad \text{and for } R >>> \hat{\rho}$$

$$r = R\left(1 + \hat{\rho}^2/R^2 - 2\hat{\rho}/R\,\cos\beta\right)^{1/2} = R\left(1 - \frac{2\hat{\rho}\cos\beta}{R}\right)^{1/2} = R\left(1 - \frac{2\hat{\rho}\cos\beta}{2R} + \cdots\right)$$

$$= R - \hat{\rho}\cos\beta = R - \hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)$$

Differential element of area in area 1 $\equiv da = \hat{\rho}\, d\hat{\rho}\, d\hat{\varphi}$

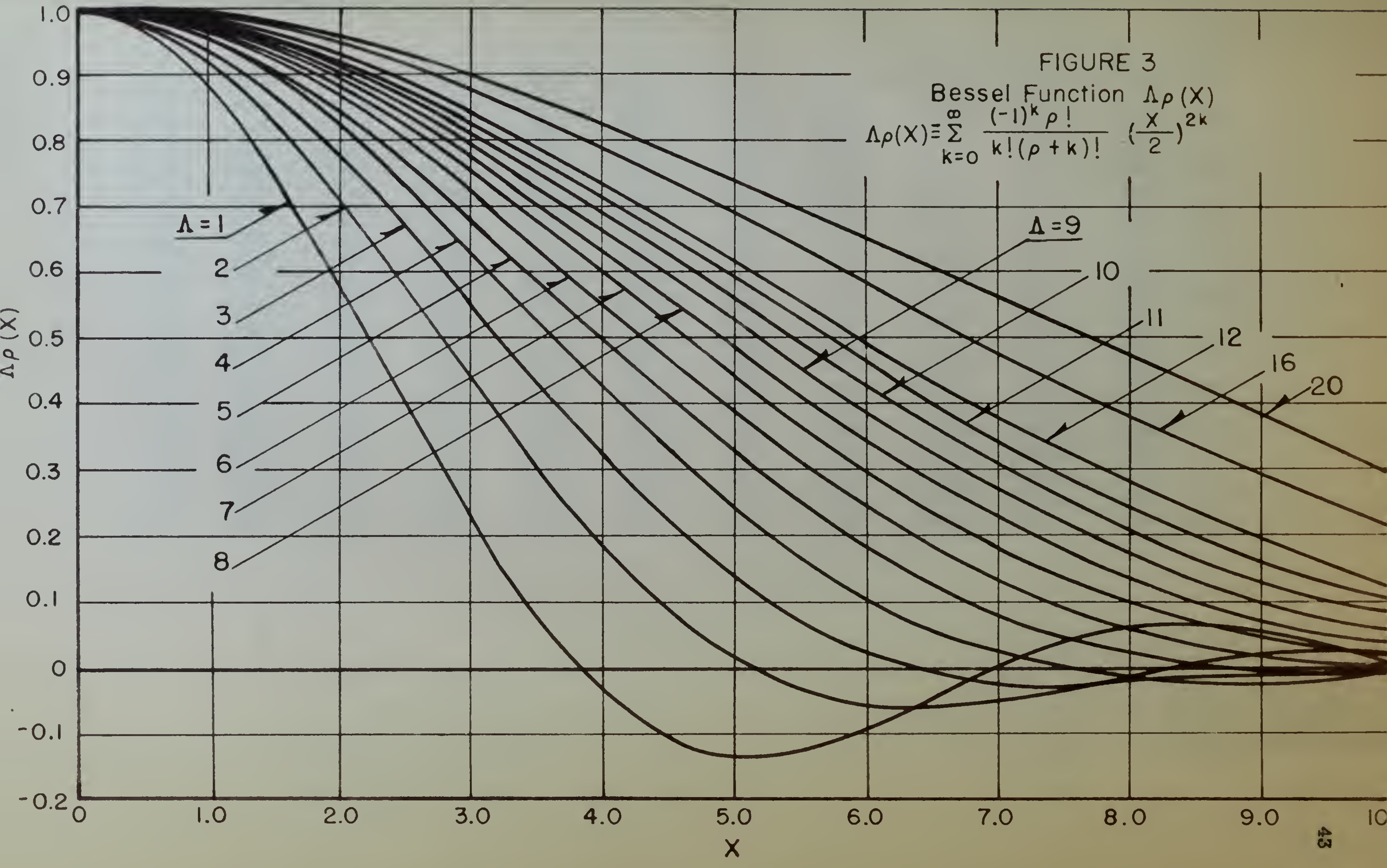

FIGURE 3

Bessel Function $\Lambda\rho(X)$

$$\Lambda\rho(X) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k \rho!}{k!(\rho+k)!} \left(\frac{X}{2}\right)^{2k}$$

Table I. Values of $\Lambda_p(x)$; $0 < p < 20$

Refer to Fig. 3

$x \backslash p$	1	2	3	4	5	6	7	8	9	10	p / x
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.0
.5	96907	97933	98447	98756	98963	99111	99221	99308	99377	99433	.5
1.0	88010	91923	93904	95103	95907	96484	96918	97257	97528	97751	1.0
1.5	74392	82520	86704	89263	90993	92241	93184	93923	94517	95005	
2.0	57672	70567	77366	81590	84476	86575	88172	89428	90442	91278	2.0
2.5	39768	57096	66540	72531	76684	79737	82079	83933	85438	86684	
3.0	22604	43308	54944	62594	67996	72021	75140	77629	79663	81357	3.0
3.5	07850	29951	43300	52306	58813	63743	67611	70729	73296	75447	
4.0	03302	18206	32263	42169	49532	55224	59756	63452	66523	69117	4.0
4.5	10269	08606	22371	32628	40520	46767	51831	56017	59535	62532	
5.0	13103	01490	14010	24037	33089	38648	44076	48634	52513	55851	5.0
5.5	12416	03103	07389	16648	24486	31094	36697	41489	45625	49227	
6.0	09223	05397	02550	10597	17881	24280	29864	34741	39021	42796	6.0
6.5	04734	05821	00618	05912	12363	18324	23703	28519	32823	36677	
7.0	00134	04291	02345	02524	07949	13285	18298	22913	27128	30965	7.0
7.5	03607	03275	02936	00289	04587	09169	13685	17982	22003	25733	
8.0	05866	01412	02729	00988	02177	05934	09862	13748	17486	21030	8.0
8.5	06426	00247	02053	01528	00581	03503	06794	10203	13591	16880	
9.0	05451	01431	01191	01554	00358	01772	04417	07315	10305	13289	9.0
9.5	03395	02020	00366	01269	00801	00623	02649	05030	07598	10242	
10.0	00869	02037	00280	00843	00899	00067	01398	03281	05423	07710	10.0

Table I (Continued)

x \ p	11	12	13	14	15	16	17	18	19	20	p / x
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.0
0.5	99480	99520	99555	99584	99610	99633	99653	99672	99689	99703	0.5
1.0	97937	98094	98229	98346	98449	98540	98620	98692	98757	98816	1.0
1.5	95413	95759	96057	96315	96542	96742	96921	97081	97225	97355	1.5
2.0	91980	92576	93090	93538	93930	94278	94588	94866	95117	95345	2.0
2.5	87733	88629	89402	90077	90670	91197	91667	92090	92471	92818	2.5
3.0	82789	84016	85079	86010	86830	87560	88213	88801	89333	89816	3.0
3.5	77277	78852	80222	81425	82489	83439	84290	85058	85754	86388	3.5
4.0	71337	73259	74939	76419	77734	78910	79968	80925	81794	82588	4.0
4.5	65115	67366	69343	71095	72657	74059	75324	76471	77517	78473	4.5
5.0	58753	61299	63551	65555	67351	68970	70436	71769	72988	74106	5.0
5.5	52389	55184	57674	59903	61911	63729	65382	66892	68276	69550	5.5
6.0	46147	49136	51818	54237	56427	58421	60242	61911	63447	64866	6.0
6.5	40139	43259	46082	48647	50985	53124	55088	56897	58568	60116	6.5
7.0	34459	37643	40552	43215	45661	47912	49991	51914	53699	55358	7.0
7.5	29181	32363	35300	38014	40525	42852	45013	47024	48898	50649	7.5
8.0	24360	27477	30387	33102	35635	38001	40211	42280	44218	46037	8.0
8.5	20030	23024	25855	28525	31039	33405	35632	37729	39705	41568	8.5
9.0	16208	19029	21735	24317	26772	29103	31314	33410	35397	37280	9.0
9.5	12891	15501	18042	20498	22860	25124	27288	29355	31326	33206	9.5
10.0	10065	12433	14779	17079	19317	21483	23574	25585	27517	29372	10.0

TABLE II

Powers of $(1 - Z^2)^n$ $0 < n < 20$

Z	$(1 - Z^2)$	$(1 - Z^2)^2$	$(1 - Z^2)^3$	$(1 - Z^2)^4$	$(1 - Z^2)^5$	$(1 - Z^2)^6$
0	1.0	1.0	1.0	1.0	1.0	1.0
.1	.99	.9801	.970299	.96059601	$.9509900\underline{4}$	$.9414801\underline{4}$
.2	.96	.9216	.884736	.84934656	$.8153726\underline{9}$	$.7827577\underline{8}$
.3	.91	.8281	.753571	.68574961	$.6240321\underline{4}$	$.5678692\underline{5}$
.4	.84	.7056	.592704	.49787136	$.4182119\underline{4}$	$.3512980\underline{3}$
.5	.75	.5625	.421875	.31640625	$.2373046\underline{8}$	$.1779785\underline{1}$
.6	.64	.4096	.262144	.16777216	$.1073741\underline{8}$	$.0687194\underline{7}$
.7	.51	.2601	.132651	.06765201	$.0345025\underline{2}$	$.0175962\underline{8}$
.8	.36	.1296	.046656	.01679616	$.0060466\underline{1}$	$.0021767\underline{8}$
.9	.19	.0361	.006859	.00130321	$.0002476\underline{0}$	$.0000470\underline{4}$
1.0	0.0	0.0	0.0	0.0	0.0	0.0

Z	$(1 - Z^2)^7$	$(1 - Z^2)^8$	$(1 - Z^2)^9$	$(1 - Z^2)^{10}$	$(1 - Z^2)^{11}$	$(1 - Z^2)^{12}$
0	1.0	1.0	1.0	1.0	1.0	1.0
.1	$.9320653\underline{4}$	$.9227446\underline{9}$	$.9135172\underline{4}$	$.9043820\underline{6}$	$.8953382\underline{4}$	$.8863848\underline{6}$
.2	$.7514474\underline{7}$	$.7213895\underline{7}$	$.6925339\underline{8}$	$.6648326\underline{2}$	$.6382393\underline{1}$	$.6127097\underline{3}$
.3	$.5167610\underline{1}$	$.4702525\underline{1}$	$.4279297\underline{8}$	$.3894160\underline{9}$	$.3543686\underline{4}$	$.3224754\underline{6}$
.4	$.2950903\underline{4}$	$.2478758\underline{8}$	$.2082157\underline{3}$	$.1749012\underline{1}$	$.1469170\underline{1}$	$.1234102\underline{8}$
.5	$.1334838\underline{8}$	$.1001129\underline{1}$	$.0750846\underline{8}$	$.0563135\underline{1}$	$.0422351\underline{3}$	$.0316763\underline{4}$
.6	$.0439804\underline{6}$	$.0281474\underline{9}$	$.0180143\underline{9}$	$.0115292\underline{0}$	$.0073786\underline{8}$	$.0047223\underline{5}$
.7	$.0089741\underline{0}$	$.0045767\underline{9}$	$.0023341\underline{6}$	$.0011904\underline{2}$	$.0006071\underline{1}$	$.0003096\underline{2}$
.8	$.0007836\underline{4}$	$.0002821\underline{1}$	$.0001015\underline{5}$	$.0000365\underline{5}$	$.0000131\underline{5}$	$.0000047\underline{3}$
.9	$.0000089\underline{3}$	$.0000016\underline{9}$	$.0000003\underline{2}$	$.0000000\underline{6}$	$.0000000\underline{1}$	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0

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TABLE II (Continued)

z	$(1 - z^2)^{13}$	$(1 - z^2)^{14}$	$(1 - z^2)^{15}$	$(1 - z^2)^{16}$	$(1 - z^2)^{17}$	$(1 - z^2)^{18}$	$(1 - z^2)^{19}$
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
.1	$.8775210\underline{0}$	$.8687457\underline{9}$	$.8600583\underline{3}$	$.8514577\underline{4}$	$.8429431\underline{7}$	$.8345137\underline{4}$	$.8261685\underline{9}$
.2	$.5882013\underline{4}$	$.5646732\underline{8}$	$.5420863\underline{4}$	$.5204028\underline{8}$	$.4995867\underline{6}$	$.4796032\underline{8}$	$.4604191\underline{4}$
.3	$.2934526\underline{6}$	$.2670419\underline{2}$	$.2430081\underline{4}$	$.2211374\underline{0}$	$.2012350\underline{3}$	$.1831238\underline{7}$	$.1666427\underline{2}$
.4	$.1036646\underline{3}$	$.0870782\underline{8}$	$.0731457\underline{5}$	$.0614424\underline{3}$	$.0516116\underline{4}$	$.0433537\underline{7}$	$.0364171\underline{6}$
.5	$.0237572\underline{5}$	$.0178179\underline{3}$	$.0133634\underline{4}$	$.0100225\underline{8}$	$.0075169\underline{3}$	$.0056376\underline{9}$	$.0042282\underline{6}$
.6	$.0030223\underline{0}$	$.0019342\underline{7}$	$.0012379\underline{3}$	$.0007922\underline{7}$	$.0005070\underline{5}$	$.0003245\underline{1}$	$.0002076\underline{9}$
.7	$.0001579\underline{0}$	$.0000805\underline{2}$	$.0000410\underline{6}$	$.0000209\underline{4}$	$.0000106\underline{7}$	$.0000054\underline{4}$	$.0000027\underline{7}$
.8	$.0000017\underline{0}$	$.0000006\underline{1}$	$.0000002\underline{1}$	$.0000000\underline{7}$	$.0000000\underline{2}$	0.0	0.0
.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

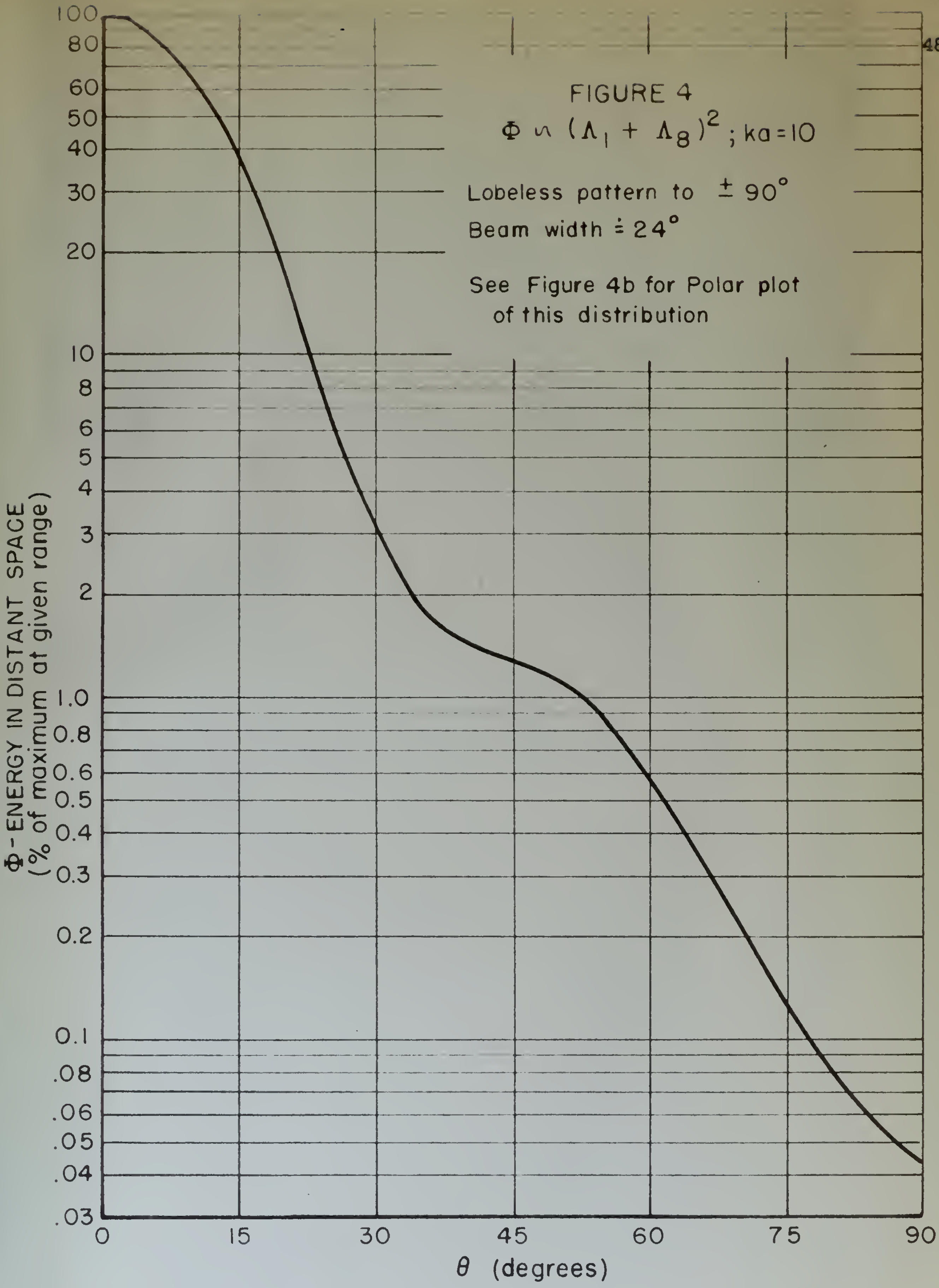

FIGURE 4

$\Phi \sim (\Lambda_1 + \Lambda_8)^2$; $ka = 10$

Lobeless pattern to ± 90°

Beam width ≐ 24°

See Figure 4b for Polar plot of this distribution

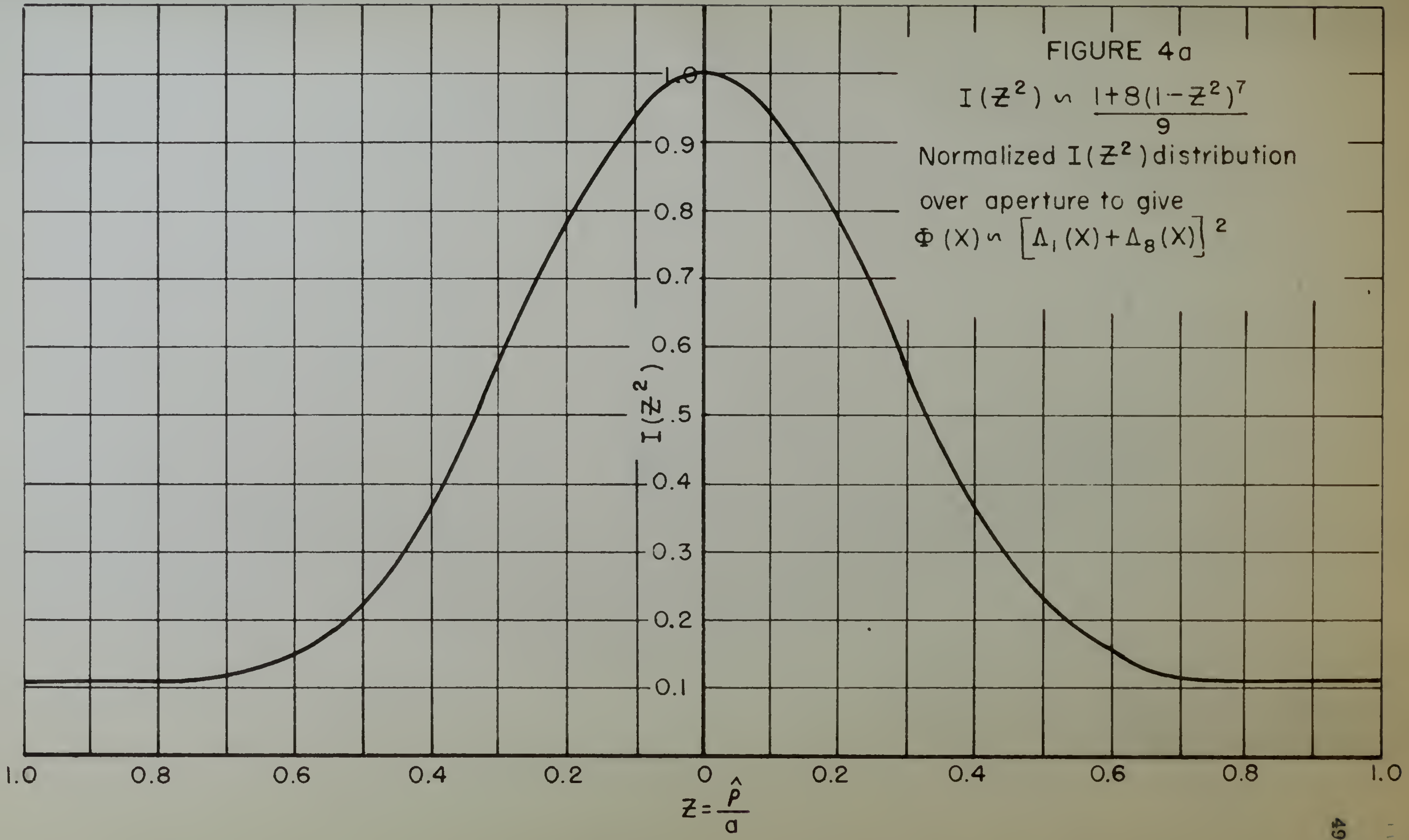

FIGURE 4a

$I(Z^2) \backsim \frac{1+8(1-Z^2)^7}{9}$

Normalized $I(Z^2)$ distribution over aperture to give

$\Phi(X) \backsim \left[\Lambda_1(X)+\Lambda_8(X)\right]^2$

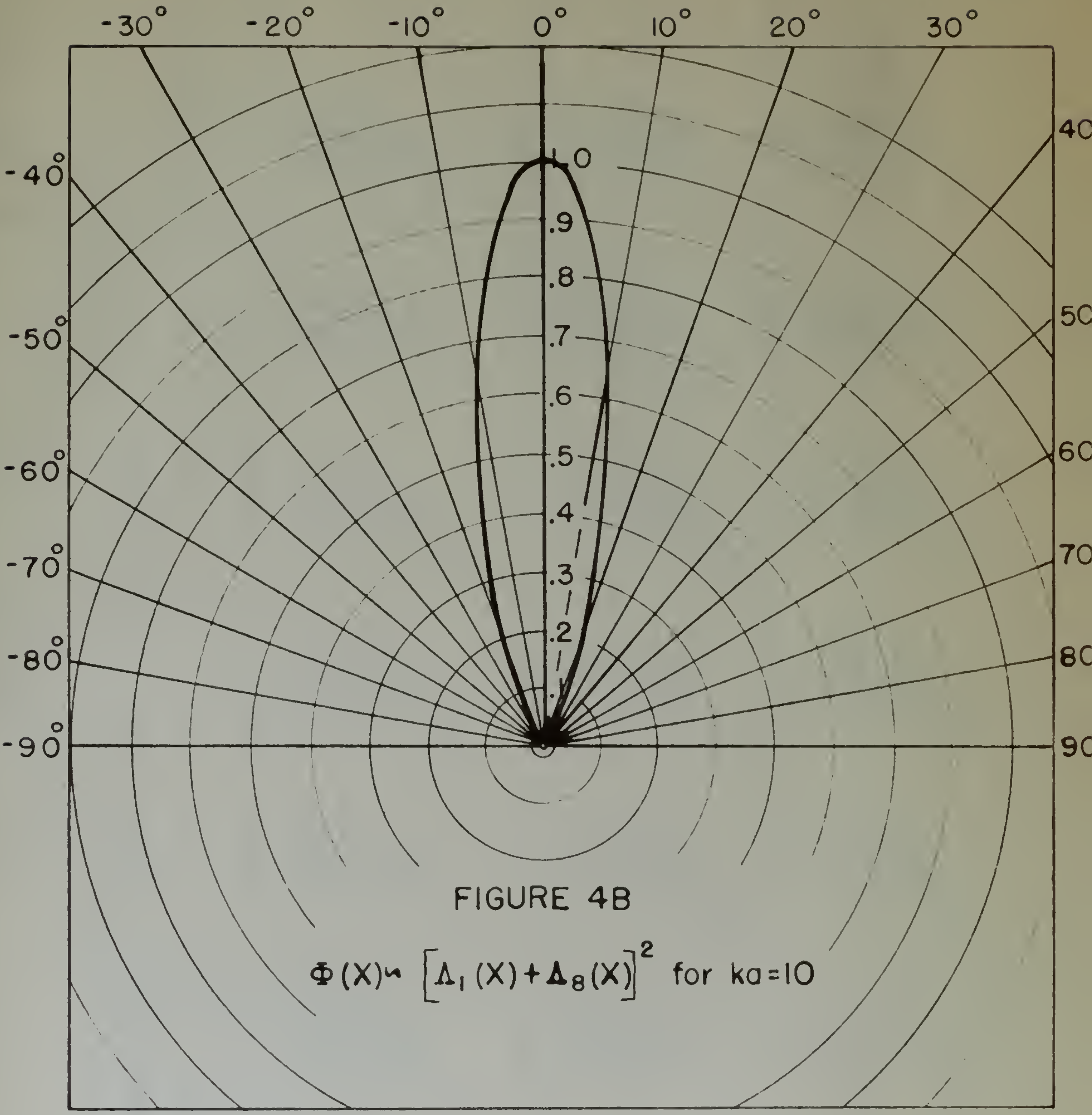

FIGURE 4B

$\Phi(X) \sim \left[\Lambda_1(X) + \Delta_8(X)\right]^2$ for ka=10

TABLE III

(Reference Figs. 4 and 4a)

Values of $\Phi(x) \sim \left[\Lambda_1(x) + \Lambda_8(x)\right]$ and corresponding $I(Z^2)$

x	0	.5	1.0	1.5	2.0	2.5	3.0
Λ_1	1	.96907	.88010	.74392	.57672	.39768	.22604
Λ_8	1	.99308	.97257	.93923	.89428	.83933	.77629
$E(x)$	2	1.96215	1.85267	1.68315	1.47100	1.23701	1.00233
$E_n(x)$	1	.98108	.92634	.84158	.73550	.61850	.50116
$\Phi(x)$	1	.962	.858	.710	.540	.384	.252
$\theta°$	0°	2.9	5.7	8.6	11.5	14.5	17.5

x	3.5	4.0	4.5	5.0	5.5	6.0	6.5
Λ_1	.07850	-.03302	-.10269	-.13103	-.12416	-.09223	-.04734
Λ_8	.70729	.63452	.56017	.48634	.41489	.34741	.28519
$E(x)$	.78579	.60150	.45748	.35531	.29073	.25518	.23785
$E_n(x)$	.39290	.30075	.22874	.17766	.14536	.12759	.11892
$\Phi(x)$	.154	.0902	.0524	.0315	.0211	.0162	.0141
$\theta°$	20.5	23.6	26.7	30.0	33.4	36.9	40.5

x	7.0	7.5	8.0	8.5	9.0	9.5	10.0
Λ_1	-.00134	+.03607	.05866	.06426	.05451	.03395	.00869
Λ_8	.22913	.17982	.13748	.10203	.07315	.05030	.03281
$E(x)$	.22779	.21589	.19614	.16629	.12766	.08425	.04150
$E_n(x)$	.11390	.10794	.09857	.08314	.06383	.04212	.02075
$\Phi(x)$	.0129	.0116	.0097	.0069	.00407	.00177	.00043
$\theta°$	44.4	48.6	53.1	58.2	64.2	71.8	90.0

TABLE III (Continued)

$$I(Z)^2 \sim 1+8\,(1-Z^2)^7$$

$I(Z^2) \diagdown Z$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
Constant	1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.
$8(1-Z^2)^7$	8.00	7.448	6.016	4.144	2.360	1.072	.352	.072	.008	-	0
$I(Z^2)$	9.000	8.448	7.016	5.144	3.360	2.072	1.352	1.072	1.008	1.000	1.000
$I_m(Z^2)$	1.000	.938	.780	.571	.373	.230	.152	.119	.112	.111	.111

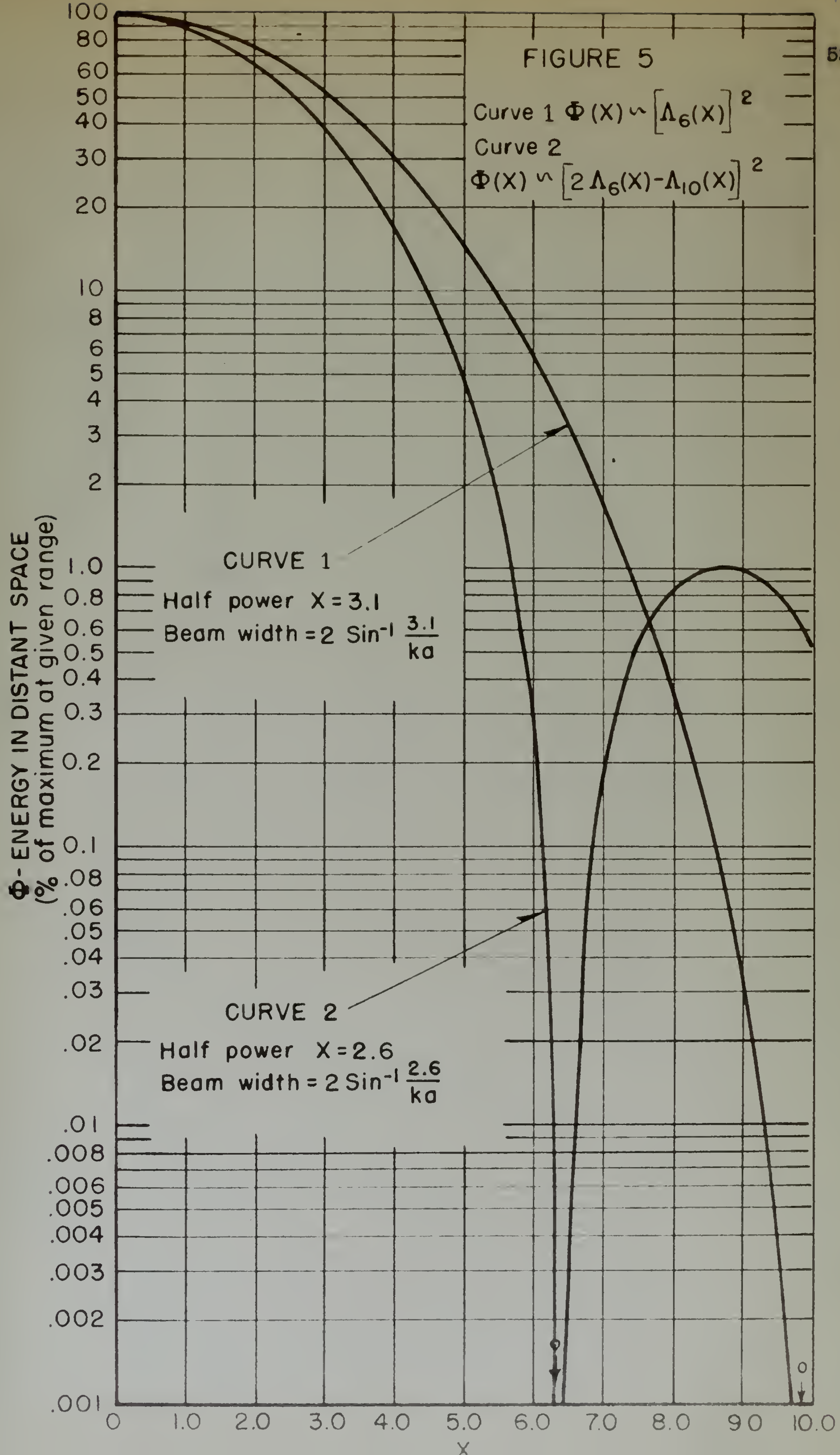

FIGURE 5

Curve 1 $\Phi(X) \sim \left[\Lambda_6(X)\right]^2$
Curve 2
$\Phi(X) \sim \left[2\Lambda_6(X)-\Lambda_{10}(X)\right]^2$
CURVE 1
Half power X = 3.1
Beam width = 2 Sin^{-1} $\frac{3.1}{ka}$
CURVE 2
Half power X = 2.6
Beam width = 2 Sin^{-1} $\frac{2.6}{ka}$
Φ - ENERGY IN DISTANT SPACE
(% of maximum at given range)
100
80
60
50
40
30
20
10
8
6
5
4
3
2
1.0
0.8
0.6
0.5
0.4
0.3
0.2
0.1
.08
.06
.05
.04
.03
.02
.01
.008
.006
.005
.004
.003
.002
.001
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
X
0
0

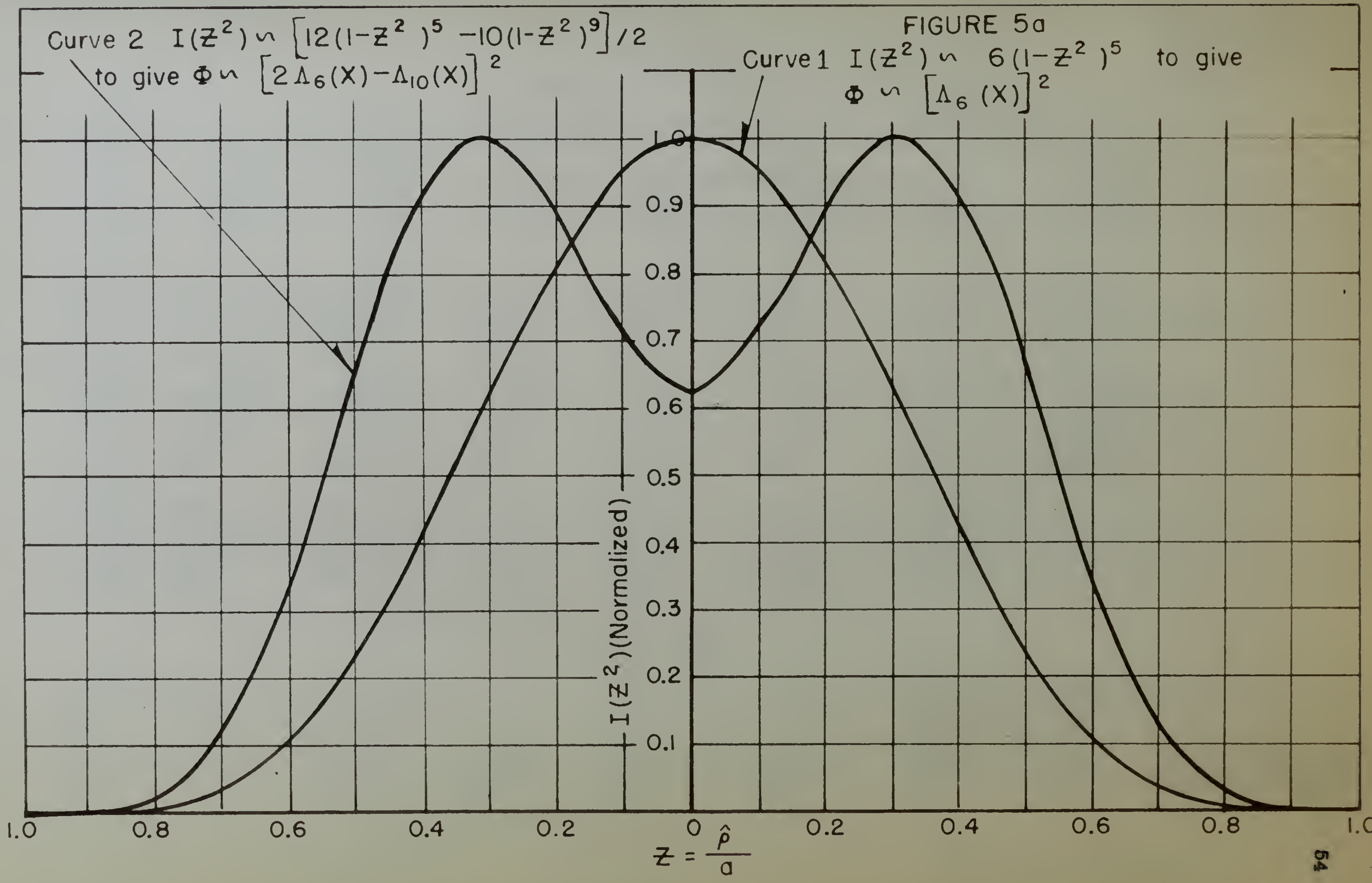
FIGURE 5a

Curve 2 $I(Z^2) \sim \left[12(1-Z^2)^5 - 10(1-Z^2)^9\right]/2$
to give $\Phi \sim \left[2\Lambda_6(X) - \Lambda_{10}(X)\right]^2$
Curve 1 $I(Z^2) \sim 6(1-Z^2)^5$ to give
$\Phi \sim \left[\Lambda_6(X)\right]^2$
$I(Z^2)$ (Normalized)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
1.0
0.8
0.6
0.4
0.2
0
0.2
0.4
0.6
0.8
1.0
$Z = \frac{\hat{\rho}}{a}$

TABLE IV

(Reference Fig. 5)

a) Values of $\Phi(x) \sim (\Lambda_6(x))^2$

b) Values of $\Phi(x) \sim \left[2\Lambda_6(x) - \Lambda_{10}(x)\right]^2$

c) Values of $I(z^2)$ corresponding to (a)

d) Values of $I(z^2)$ corresponding to (b)

-a-

$\Lambda \diagdown x$	0	.5	1.0	1.5	2.0	2.5	3.0
Λ_6	1.0	.99111	.96484	.92241	.86575	.79737	.72021
$\Phi(x)$	1.0	.980	.930	.850	.750	.636	.519

$\Lambda \diagdown x$	3.5	4.0	4.5	5.0	5.5	6.0	6.5
Λ_6	.63743	.55224	.46767	.38648	.31094	.24280	.18324
$\Phi(x)$	.406	.305	.219	.149	.0972	.0590	.0336

$\Lambda \diagdown x$	7.0	7.5	8.0	8.5	9.0	9.5	10.0
Λ_6	.13285	.09169	.05934	.03503	.01772	.00623	-.00067
$\Phi(x)$	.0176	.0084	.00351	.00123	.000314	.0000389	-

-b-

$\Lambda \diagdown x$	0	.5	1.0	1.5	2.0	2.5	3.0
$2\Lambda_6$	2.0	1.98222	1.92968	1.84482	1.73150	1.59474	1.4402
$-\Lambda_{10}$	1.0	.99433	.97751	.95005	.91278	.86684	.81357
$E_m(x)$	1.0	.98789	.95217	.89477	.81872	.72790	.62685
$\Phi(x)$	1.0	.976	.908	.800	.670	.530	.394

Continued following page

TABLE IV (Continued)

-b- (continued)

$\Lambda \backslash \chi$	3.5	4.0	4.5	5.0	5.5	6.0	6.5
$2\Lambda_6$	1.27486	1.10448	.93534	.77296	.62188	.48560	.36248
$-\Lambda_{10}$	.75447	.69117	.62532	.55851	.49227	.42796	.36677
$E_m(\chi)$	.52039	.41331	.31002	.21445	.12961	.05764	-.00429
$\Phi(\chi)$	.270	.171	.096	.046	.0168	.00331	.0000184

$\Lambda \backslash \chi$	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$2\Lambda_6$	.26570	.18338	.11868	.07006	.03544	.01246	-.00134
$-\Lambda_{10}$	.30965	.25733	.21030	.16880	.13289	.10242	.07710
$E_m(\chi)$	-.04395	-.07395	-.09162	-.09874	-.09745	-.08996	-.07844
$\Phi(\chi)$	.00193	.00546	.0084	.00973	.0095	.0081	.00615

$I(Z^2) \sim 6(1-Z^2)^5$

-c-

$\backslash Z$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$I_m(Z^2)$	1	.951	.815	.625	.418	.237	.107	.035	.006	-	0

$I(Z^2) \sim 12(1-Z^2)^5 - 10(1-Z^2)^9$

-d-

$\backslash Z$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$12(1-Z^2)^5$	12.000	11.41	9.78	7.50	5.025	2.84	1.284	0.420	.072	-	0
$-10(1-Z^2)^9$	10.000	9.13	6.93	4.30	2.080	.75	.180	.020	-	-	0
$I(Z^2)$	2.000	2.28	2.85	3.20	2.945	2.09	1.104	.40	.072	-	0
$I_m(Z^2)$	0.625	.712	.890	1.00	.92	.654	.345	.125	.0225	-	0

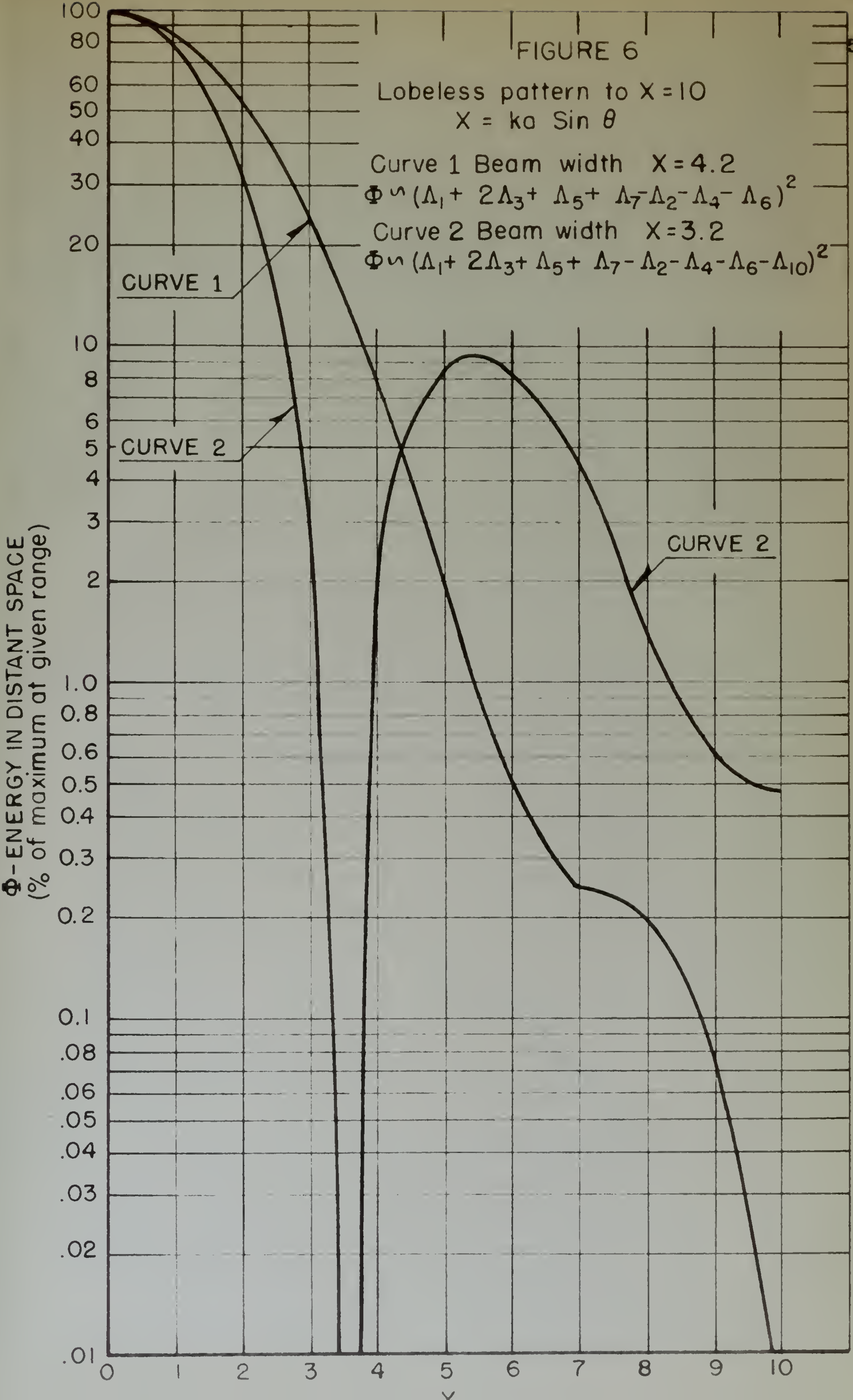

FIGURE 6

Lobeless pattern to X = 10

$X = ka \sin\theta$

Curve 1 Beam width X = 4.2

$\Phi \backsim (\Lambda_1 + 2\Lambda_3 + \Lambda_5 + \Lambda_7 - \Lambda_2 - \Lambda_4 - \Lambda_6)^2$

Curve 2 Beam width X = 3.2

$\Phi \backsim (\Lambda_1 + 2\Lambda_3 + \Lambda_5 + \Lambda_7 - \Lambda_2 - \Lambda_4 - \Lambda_6 - \Lambda_{10})^2$

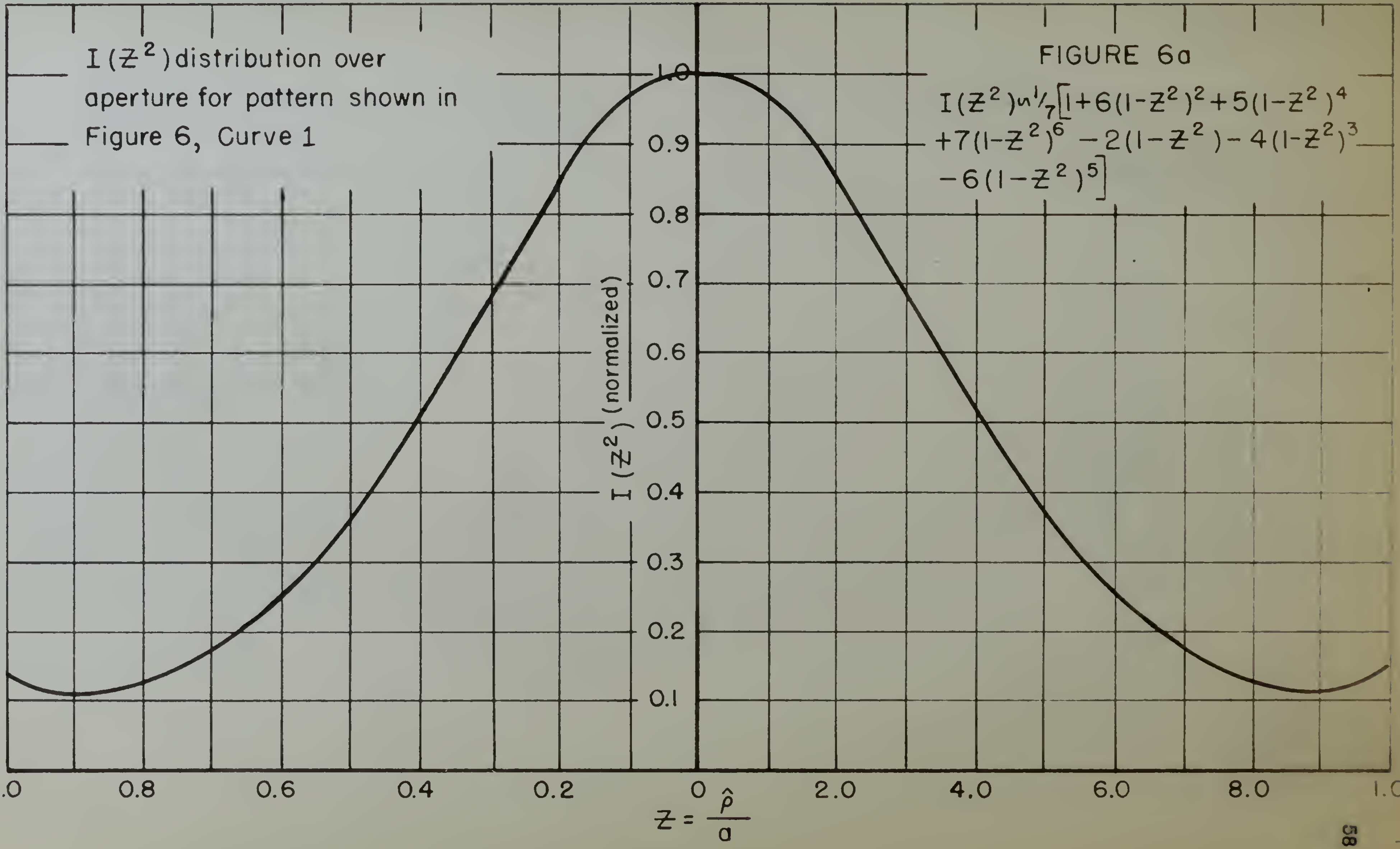

FIGURE 6a

$I(Z^2) \propto \frac{1}{7}\left[1+6(1-Z^2)^2+5(1-Z^2)^4 +7(1-Z^2)^6 -2(1-Z^2)-4(1-Z^2)^3 -6(1-Z^2)^5\right]$

$I(Z^2)$ distribution over aperture for pattern shown in Figure 6, Curve 1

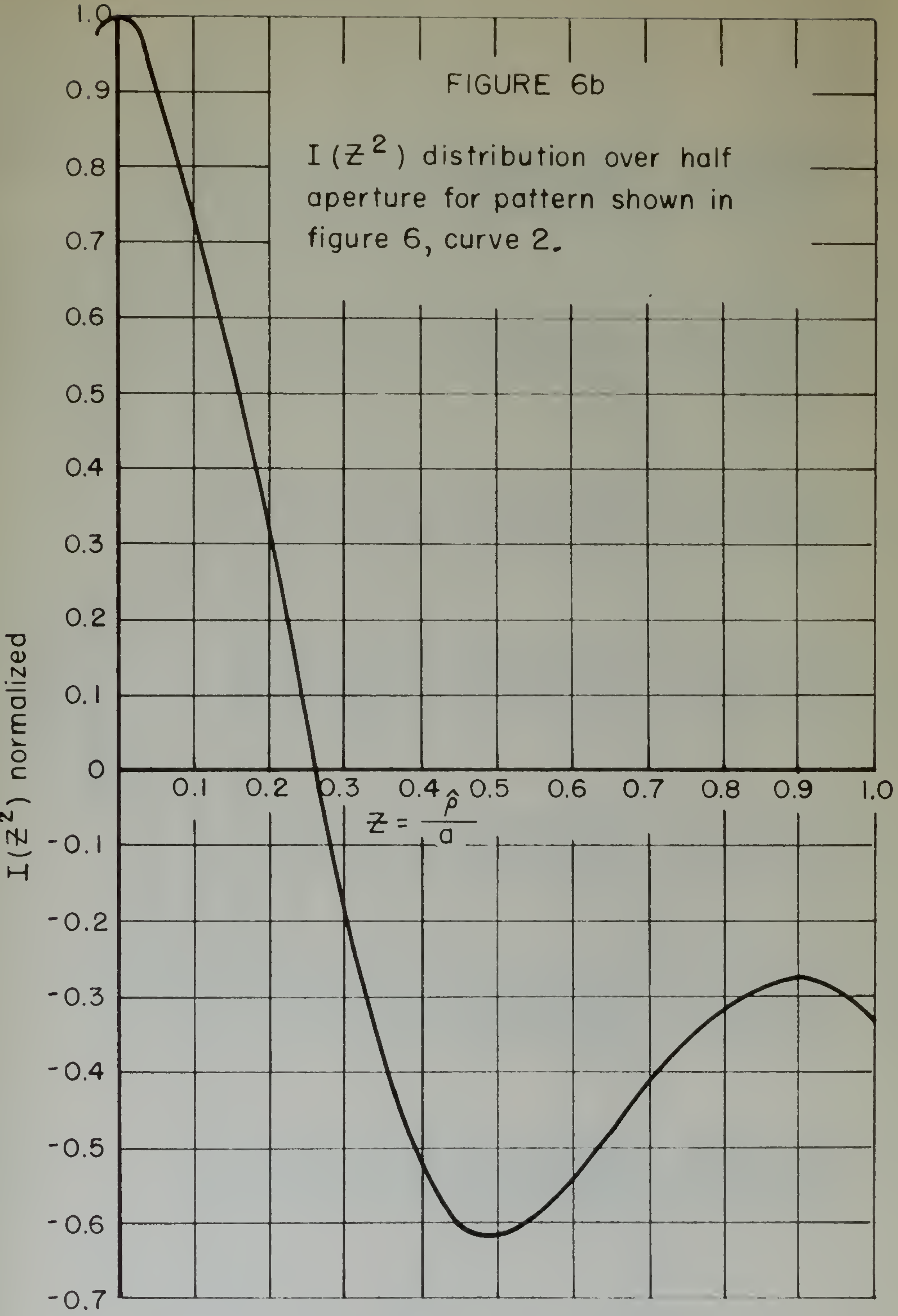

FIGURE 6b

$I(Z^2)$ distribution over half aperture for pattern shown in figure 6, curve 2.

TABLE V

(Reference Fig. 6)

a) Values for $\Phi(x) \sim \left[\Lambda_1(x)+2\Lambda_3(x)+\Lambda_5(x)+\Lambda_7(x)-\Lambda_2(x)-\Lambda_4(x)-\Lambda_6(x)\right]^2$

b) Values for $\Phi(x) \sim \left[\Lambda_1(x)+2\Lambda_3(x)+\Lambda_5(x)+\Lambda_7(x)-\Lambda_2(x)-\Lambda_4(x)-\Lambda_6(x)-\Lambda_{10}(x)\right]^2$

c) Values for $I(Z^2)$ corresponding to (a)

d) Values for $I(Z^2)$ corresponding to (b)

-a-

$E(x) \sim \Lambda_1 + 2\Lambda_3 + \Lambda_5 + \Lambda_7 - \Lambda_2 - \Lambda_4 - \Lambda_6$

$\Lambda \diagdown x$	0	.5	1.0	1.5	2.0	2.5	3.0
1	1.00000	.96907	.88010	.74392	.57672	.39768	.22604
(2)3	2.00000	1.96894	1.87808	1.73408	1.54732	1.33080	1.09888
5	1.00000	.98263	.95907	.90993	.84476	.76684	.67996
7	1.00000	.99221	.96918	.93184	.88172	.82079	.75140
Total	5.00000	4.91985	4.68643	4.31977	3.85052	3.31611	2.75628
-(2+4+6)	3.00000	2.95800	2.83510	2.64024	2.38732	2.09364	1.77823
E(X)	2.00000	1.96185	1.85133	1.67953	1.46320	1.22247	1.97805
E(X)n	1.00000	.98093	.92567	.83977	.73160	.61124	.48903
Φ(X)	1.00000	.960	.858	.704	.535	.375	.239
2	1.00000	.97933	.91923	.82520	.70567	.57096	.43208
4	1.00000	.98756	.95103	.89263	.81590	.72331	.62594
6	1.00000	.99111	.96484	.92241	.86575	.79737	.72021
Total	3.00000	2.95800	2.83510	2.64024	2.38732	2.09364	1.77823

Continued following page

TABLE V (Page 2)

-a- (continued)

Λ \ x	3.5	4.0	4.5	5.0	5.5	6.0	6.5
1	.07850	-.03202	-.10269	-.13103	-.12416	-.09223	-.04734
(2)3	.86600	.64426	.44742	.28020	.14778	+.05100	-.01236
5	.58813	.49532	.40520	.32089	.24486	.17881	.12363
7	.67611	.59756	.51813	.44076	.36697	.29864	.23703
Total	2.20874	1.70512	1.26806	.91082	.63545	.43622	.30096
-(2+4+6)	1.46000	1.15599	.88001	.64175	.44639	.29480	.18415
E(X)	.74874	.54913	.38805	.26907	.18906	.14142	.11681
(X)n	.37437	.27457	.19403	.13454	.09453	.07071	.05841
(X)	.140	.0758	.0376	.0181	.00892	.00500	.00341
2	.29951	.18206	.08606	+.01490	-.03103	-.05597	-.05821
4	.52306	.42169	.32628	.24037	.16648	.10597	.05912
6	.63743	.55224	.46767	.38648	.31094	.24480	.18324
Total	1.46000	1.15599	.88001	.64175	.44639	.29480	.18415

Continued following page

TABLE V (Page 3)

-a- (continued)

$\Lambda \backslash x$	7.0	7.5	8.0	8.5	9.0	9.5	1.0
1	-.00134	+.03607	.05866	.06426	.05431	.03395	.00869
(2)3	-.04690	-.05872	-.05458	-.04106	-.02382	-.00732	+.00560
5	.07949	.04387	.02177	.00581	-.00358	-.00801	-.00899
7	.18298	.13685	.09862	.06794	.04417	.02649	.01398
Total	.21423	.16007	.12447	.09695	.07128	.04511	.01928
-(2+4+6)	.11518	.06183	.03534	.02222	.01649	.01374	.00127
M(X)	.09905	.09824	.08913	.07473	.05479	.03137	.00801

E(X)n	.04958	.04912	.04457	.03727	.02729	.01569	.00400
$\Phi(x)$	.00246	.00231	.00199	.00140	.000744	.000246	.000016
2	-.04291	-.03275	-.01412	+.00247	.01431	.02020	.02037
4	.02524	+.00289	-.00988	-.01528	-.01554	-.01269	-.00843
6	.17285	.09169	.05934	.03503	.01772	+.00623	-.00067
Total	.11518	.06183	.03534	.02222	.01649	.01374	.01127

-b-

$\backslash x$	0	.5	1.0	1.5	2.0	2.5	3.0
$E_1(x)$	2.00000	1.96185	1.85133	1.67953	1.46320	1.22247	.97805
$-\Lambda_{10}(x)$	1.00000	.99433	.97751	.95005	.91278	.86684	.81357
$E_2(x)$	1.00000	.96752	.87382	.72948	.55042	.35563	.16448
$\Phi_2(x)$	1.00000	.935	.763	.531	.3025	.127	.027

Continued following page

TABLE V (Page 4)

-b- (continued)

x	3.5	4.0	4.5	5.0	5.5	6.0	6.5
$E_1(x)$	.74874	.54913	.38805	.26907	.18906	.14142	.11681
$-\Lambda_{10}(x)$	.75447	.69117	.62532	.55851	.49227	.42796	.36677
$E_2(x)$	-.00573	-.14204	-.23727	-.28944	-.30321	-.28654	-.24996
$\mathbb{E}_2(x)$	.0000329	.0202	.0561	.0836	.092	.082	.0625

x	7.0	7.5	8.0	8.5	9.0	9.5	10.0
$E_1(x)$	.09905	.09824	.08913	.07473	.05479	.03137	.00801
$-\Lambda_{10}(x)$	.30965	.25733	.21030	.16880	.13289	.10242	.07710
$E_2(x)$	-.21060	-.15909	-.12117	-.09407	-.07810	-.07105	-.06909
$\mathbb{E}_2(x)$	.0445	.0253	.0147	.00885	.0061	.00505	.00478

-c-

Z	0	.1	.2	.3	.4	.5	.6
Const.	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$6(1-Z^2)^2$	6.000	5.880	5.532	4.969	4.236	3.372	2.460
$5(1-Z^2)^4$	5.000	4.800	4.425	3.435	2.490	1.580	.840
$7(1-Z^2)^6$	7.000	6.587	5.481	3.985	2.457	1.246	.483
Total +	19.000	18.267	16.258	13.386	10.183	7.198	4.783
Total -	12.000	11.446	10.350	8.590	6.620	4.610	2.970
$I(Z^2)$	7.000	6.821	5.908	4.796	3.623	2.588	1.813
$I_n(Z^2)$	1.000	0.974	.844	.685	.518	.370	.259
$2(1-Z^2)$	2.000	1.980	1.920	1.820	1.680	1.500	1.280
$4(1-Z^2)^3$	4.000	3.880	3.540	3.020	2.372	1.688	1.048
$6(1-Z^2)^5$	6.000	5.706	4.890	3.750	2.508	1.422	.642
Total +	12.000	11.446	10.350	8.590	6.560	4.610	2.970

Continued following page

TABLE V(Case 5)

-c- (continued)

Z	.7	.8	.9	1.0
Const.	1.000	1.000	1.000	1.000
$6(1-Z^2)^2$	1.560	.780	.216	
$5(1-Z^2)^4$	.340	.085	.005	
$7(1-Z^2)^6$	.126	.014	.000	
Total +	3.026	1.879	1.221	
Total -	1.762	.944	.408	1.000
$I(Z^2)$	1.264	.935	.813	1.000
$I_n(Z^2)$	.181	.134	.116	.143
$2(1-Z^2)$	1.020	.720	.380	
$4(1-Z^2)^3$	.532	.188	.028	
$6(1-Z^2)^5$	.210	.056	.000	
Total +	1.762	.944	.408	0

-d-

Z	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$I_1(Z^2)$	7.000	6.821	5.908	4.796	3.623	2.588	1.813	1.264	.935	.813	1.000
$10(1-Z^2)^9$	10.000	9.130	6.930	4.300	2.080	.750	.180	.020	-	-	-
Total	-3.000	-2.309	-1.022	+.496	1.543	1.838	1.633	1.244	.935	.813	1.000
$I_n(Z^2)$	-1.0000	-.730	-.341	+.165	+.314	+.613	.544	.415	.312	.271	.333

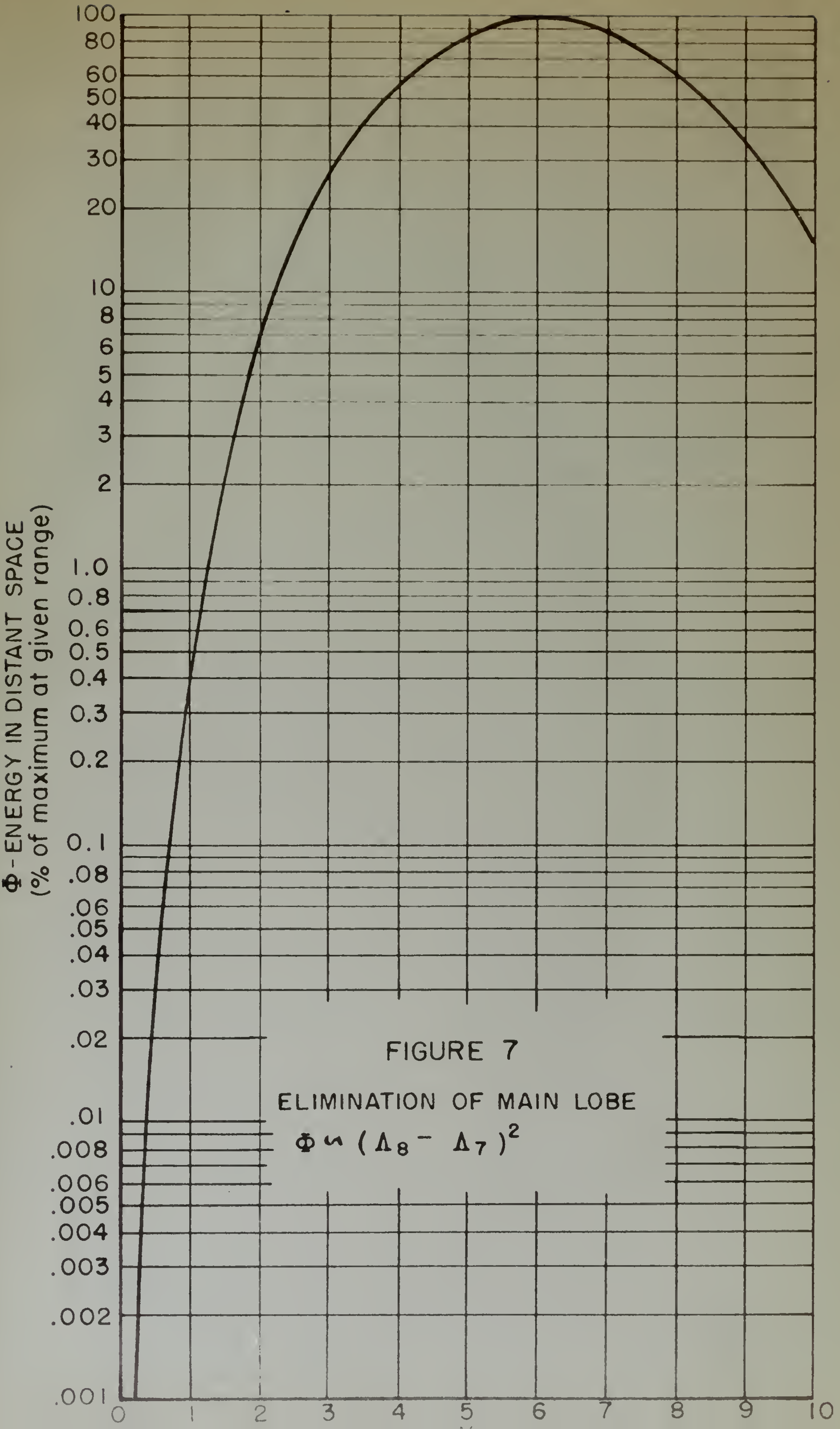

FIGURE 7

ELIMINATION OF MAIN LOBE

$\Phi \propto (\Lambda_8 - \Lambda_7)^2$

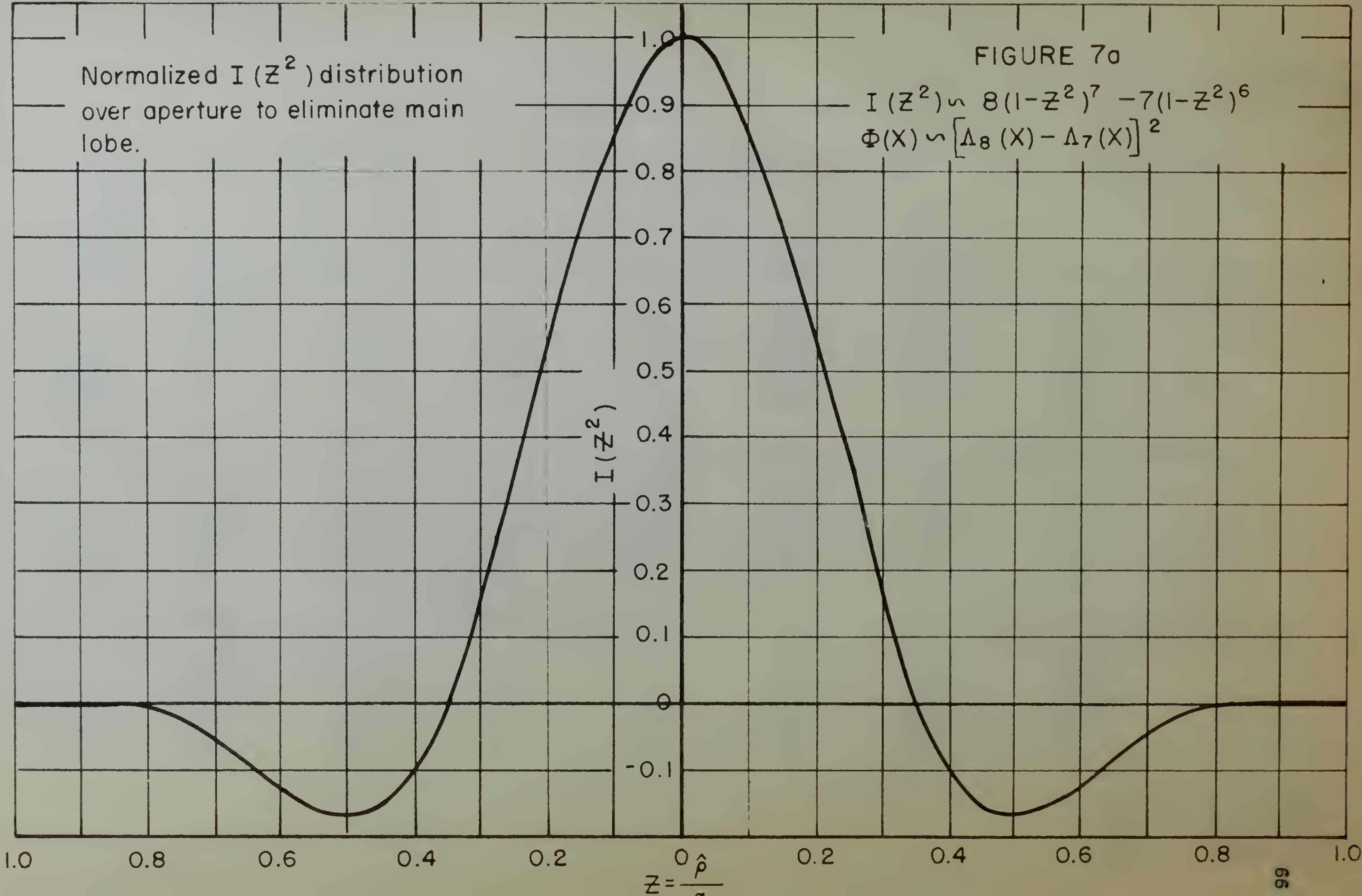

FIGURE 7a

$I(Z^2) \backsim 8(1-Z^2)^7 - 7(1-Z^2)^6$

$\Phi(X) \backsim \left[\Lambda_8(X) - \Lambda_7(X)\right]^2$

Normalized $I(Z^2)$ distribution over aperture to eliminate main lobe.

TABLE VI

(Reference Figs. 7)

a) Values for $\Phi(\chi) \backsim \left[\Lambda_8(\chi) - \Lambda_7(\chi)\right]^2$

b) Values for $I(\chi^2)$ corresponding to (a)

-a-

$$\Phi(\chi) \backsim (\Lambda_8 - \Lambda_7)^2$$

χ	0	.5	1.0	1.5	2.0	2.5	3.0
Λ 8	1.00000	.99308	.97257	.95923	.89428	.83923	.77629
Λ 7	1.00000	.99221	.96918	.93184	.88172	.82079	.75140
E(x)	0	.00087	.00339	.00739	.01256	.01854	.02489
$E_n(x)$	0	.01783	.06950	.15120	.25750	.38000	.5100
(x)	0	.000318	.00433	.0229	.066	.144	.260

χ	3.5	4.0	4.5	5.0	5.5	6.0	6.5
Λ 8	.70729	.63452	.56017	.48634	.41489	.34741	.28519
Λ 7	.67611	.59756	.51831	.44076	.36697	.29864	.23703
E(x)	.03118	.03696	.04186	.04558	.04792	.04877	.04816
$E_n(x)$	.638	.757	.857	.935	.962	1.0000	.988
(x)	.407	.572	.733	.872	.923	1.0000	.978

χ	7.0	7.5	8.0	8.5	9.0	9.5	10.0
Λ 8	.22913	.17992	.13748	.10203	.07315	.05030	.03281
Λ 7	.18298	.13685	.09862	.06794	.04417	.02649	.03198
E(x)	.04615	.04297	.03886	.03409	.02898	.02381	.01883
$E_n(x)$	.948	.881	.797	.699	.594	.489	.386
(x)	.900	.776	.634	.488	.353	.239	.149

Continued following page

TABLE VI (Continued)

-b-

$$I(Z^2) \sim 8(1-Z^2)^7 - 7(1-Z^2)^6$$

Z	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0
$8(1-Z^2)$	8	7.448	6.016	4.144	2.360	1.072	.362	.072	.008	-	0
$7(1-Z^2)$	7	6.587	5.461	3.983	2.457	1.246	.483	.126	.014	-	0
$I_n(Z^2)$	1.0000	.861	.555	.161	-.097	-.174	-.121	-.054	-.006	-	0

Appendix IV

A Special Property of Determinants

Given a set of n simultaneous equations of the form obtained by equating coefficients of Λ_m of equation (4.07) where n - 1 equations are equal to zero. We have an array of equations as follows:

$$\begin{array}{lllllllll}
(1) & 0 & = & a_0 I_0 & + a_2 I_2 & + a_4 I_4 & + a_6 I_6 & + \dots + & a_{2(n-1)} I_{(2(n-1)} \\
(2) & 0 & = & 0 & + b_2 I_2 & + b_4 I_4 & + b_6 I_6 & + \dots + & b_{2(n-1)} I_{2(n-1)} \\
(3) & 0 & = & 0 & + 0 & + c_4 I_4 & + c_6 I_6 & + \dots + & c_{2(n-1)} I_{2(n-1)} \\
\vdots & & & & & & & & \\
(j) & A & = & 0 & + 0 & + \dots & + x_{2(j-1)} I_{2(j-1)} & + \dots + & x_{2(n-1)} I_{2(n-1)} \\
\vdots & & & & & & & & \\
(n) & 0 & = & 0 & + 0 & + \dots & \dots & \dots + & z_{2(n-1)} I_{2(n-1)}
\end{array}$$

Assume first that none of the I_{2k} is zero and that all coefficients exist. We may then solve for I_{2k} $\frac{D_{2k}}{D}$ where D is the determinant of the coefficients of the I_{2k} and D_{2k} is the determinant formed by replacing the I_{2k} column by the column on the left. Then all D_{2k} for $2k > 2(j - 1)$ will be zero and all I_{2k} for $2k > 2(j - 1)$ will be zero since the first column of D_{2k} will be a linear combination of columns 2 to $2(j - 1)$ and column 2k. This reduces the number of equations in the original array to j and the j^{th} equation may then be solved for $I_{2(j - 1)}$. The other equations may then be solved to give the remaining I_{2k} for $2k < 2(j - 1)$.

As an example consider n equations for $n = 5$, $j = 3$.

$$\begin{array}{lllllll}
0 & = & a_0 J_0 & + a_2 I_2 & + a_4 I_4 & + a_6 I_6 & + a_8 I_8 \\
0 & = & 0 & + b_2 I_2 & + b_4 I_4 & + b_6 I_6 & + b_8 I_8 \\
A & = & 0 & + 0 & + c_4 I_4 & + c_6 I_6 & + c_8 I_8 \\
0 & = & 0 & + 0 & + 0 & + d_6 I_6 & + d_8 I_8 \\
0 & = & 0 & + 0 & + 0 & + 0 & + e_8 I_8
\end{array}$$

then $I_6 = \dfrac{D_6}{D}$. We assume that I_6 and I_8 exist and that $D \neq 0$.

In determinant form we have

$$\begin{array}{ccc}
\text{Column} & \begin{array}{ccccc} 0 & 2 & 4 & 6 & 8 \end{array} & \\
D_6 = & \begin{vmatrix} a_0 & a_2 & a_4 & 0 & a_8 \\ 0 & b_2 & b_4 & 0 & b_8 \\ 0 & 0 & c_4 & A & c_8 \\ 0 & 0 & 0 & 0 & d_8 \\ 0 & 0 & 0 & 0 & e_8 \end{vmatrix} & = 0
\end{array}$$

However $D_6 = 0$ since the 0 column is identical with

$$\frac{a_0}{a_2 - \dfrac{a_4 b_2}{b_4}} \left[\text{Column } 2 - \frac{b_2}{b_4} \left\{ \text{Column } 4 - \frac{c_4}{A} \ \text{Column } 6 \right\} \right]$$

Similarly D_8 0

Hence, our original assumption was incorrect and we have left j non-trivial equations which may be easily solved.

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